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# ELEMENTS

OF

# GEOMETRY

UPON

# THE INDUCTIVE METHOD.

TO WHICH IS ADDED

# AN INTRODUCTION

TO

# DESCRIPTIVE GEOMETRY.

BY JAMES HAYWARD, A. M.
Lately College Professor of Mathematics and Natural Philosophy
in Harvard University.

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# PREFACE.

In the volume now offered to the public, the author has aimed to bring the science of Elementary Geometry within limits which should adapt it to the convenience of a greater number than can afford time to acquire a competent knowledge of the subject from the treatises in common use; while, so far as this branch of elementary preparation is concerned, he should omit nothing essentially introductory to the higher geometry, or to the physico-mathematical sciences. He has endeavoured also to state the first principles of the science in a manner better suited to the apprehension of the young student, than that in which they are usually presented in elementary treatises.

Clairaut, one of the first mathematical geniuses of the eighteenth century, has said, "Though Geometry be in itself abstract, it must nevertheless be acknowledged, that the difficulties which the student encounters in the commencement of the subject, most frequently arise from the manner in which it is taught in common Elementary Treatises. These begin with a great number of definitions, postulates, axioms, and preliminary principles, which seem to promise little to interest the reader. The propositions which follow, do not fix the mind upon engaging objects; and as they are moreover difficult to be understood, it commonly happens that beginners are fatigued and dis-

gusted before they have any distinct idea of what is designed to be taught them."

In the present treatise, the author has endeavoured, as far as he was able, to remove the difficulties here complained of; by avoiding the abstract phraseology and technical forms of the books in common use; by presenting to the mind of the learner, in a manner as simple as he could, the elementary truths of the science; and developing by a process of plain and, at the same time, exact examination, those which are more involved; by following in a great degree the natural order of discovery; and aiming so to conduct the investigation of any particular fact or general principle, as that other collateral truths should unfold themselves incidentally, and thus, by the very circumstance of their being unexpected, afford pleasure to the learner, and excite in his mind a curiosity to know what will next discover itself on either side of the path he is pursuing.

This desire to know with what interesting and useful truths the further pursuit of the subject will bring him acquainted, will enable the student to perform with pleasure what is commonly (and frequently with too much truth) called his *task*, and really to do more with less fatigue. It is, moreover, a state of the mind in which the truths actually presented to it make the greatest impression; and are most likely to become a permanent part of the student's knowledge.

If the result is kept back till the learner has gone through the process of induction by which any truth is to be established, and by which also it must have been at first discovered, his mind will be prepared to perceive the precise meaning of the proposition in which this truth is enunciated; whereas the slightest misapprehension of the *nature* or the *extent* of the truth stated beforehand in the proposition, may seriously embarrass him in applying the process of proof, to say nothing of the difficulty of ridding

himself entirely of the first impression made by a proposition endowed with the oracular name of *theorem.*

Another important advantage results from thus withholding the proposition till its truth is apparent. The student will early begin to anticipate, before he arrives at the statement of the result, the truths as they unfold themselves; and finding that he has discovered them from the relations presented, he is gratified by this evidence of his own power, and is encouraged to continue his exertions. He will be constantly on the watch for new discoveries; and his studies will be a salutary discipline, not only to his understanding, but to the inventive faculty of his mind.

In preparing these Elements, the author has consulted various editions of Euclid, and several modern treatises, among which may be mentioned those of Legendre, Bézout, Reynaud, Lacroix, and Clairaut; besides the meta physical writings of Lacroix, Carnot, and Laplace. He has endeavoured to combine, especially in the earlier part of the work, the *simplicity* of Clairaut with the *certainty* of what are usually called the "severer methods." In the latter part, the general arrangement of the subject is more nearly that of Lacroix than of any other. But most of what may be considered as peculiar, whether in the arrangement or the manner of conducting the inquiries by which the truths of the science should unfold themselves to the student, has been suggested by a careful observation of the difficulties which (in the course of no inconsiderable experience as an instructer in elementary mathematics) the author saw his pupils continually encountering; and by the results of constant efforts to investigate the causes of these difficulties, and of multiplied endeavours to state the elementary principles of the subject in so simple a manner as to be apprehended by every mind. Though he may not always have succeeded, yet the results were such as to con-

vince him that the mistakes and misgivings which the young student experiences in the early part of his geometrical studies, are very uncertain indications of any peculiar want of adaptation of mind to this branch of science. So far from it, these very mistakes should be expected. The subject is new; the leading ideas are many of them expressed in terms almost exclusively appropriated to this science, and must therefore be new to the beginner; though these terms seem very definite to the metaphysical geometer, there is much uncertainty whether they will convey to the mind of the student the precise idea which the writer intended; and it will frequently happen, that the ideas which many, even of the better scholars, receive from some of the principal definitions and statements in our elementary books, require to be considerably modified or entirely changed, to adapt them to the use which is to be made of them in the subsequent part of their studies.

How much perplexity and discouragement might have been spared the learner in such cases, if the mistake could have been discovered at first. In consequence of a slight misapprehension of some leading principle in the beginning, to which every mind is liable, the learner tasks himself in vain to reconcile the subsequent reasoning and results with the first notions which he received, and which, coming with all the freshness of novelty, fixed themselves in his mind. Such mistakes might be, in a great measure, prevented, if the instructer would explain and illustrate every sentence of one or two of the first lessons before they are read by his pupils; and if, wherever any new element is introduced, he would be sure that they understand it before they proceed to apply it.

There is always danger in giving the beginner two or three pages, at first, of elementary matter in the form of distinct principles and definitions, attended by little or no explanation. Such an array of detached and unconnected truths tends only to fatigue and distract him.

To prevent, in some degree, this liability to misconception and embarrassment, the definitions and principles, in the present treatise, are given only as they are to be used in the course of the investigation, either as principal or auxiliary truths.

For some of the elements new definitions have been adopted, which, it is believed, convey to the mind of the learner more accurate notions of the things defined, and very much simplify the processes into which they enter. "A good definition should contain an enumeration of certain simple characteristic attributes of the thing defined, by which it may be clearly distinguished from all other things of a like kind. In this respect the definitions of mathematics are, in general, peculiarly happy. They usually contain some one simple but characteristic property of the thing in question, from which all its other properties may be readily and legitimately deduced."*

In conformity with the spirit of the remark here quoted, the three following definitions are adopted.

*A straight line is one which has the same direction throughout its whole extent.*

*A plane angle is the inclination of two straight lines to each other.*

*Parallel lines are straight lines which have the same direction in space.*

The simple characteristic property of a straight line, that is, its *straightness*, by which it is distinguished from all other lines, is its *identity of direction* in every part. The definition given in several elementary books, namely, that *a straight line is the shortest way from one point to another*, is a proposition which carries to the mind the fullest conviction of its truth; and whether a simple principle or a

* Prof. Elliot of Aberdeen.

deducible truth, it is equally a suitable element of geometrical reasoning. But this identity of direction in all its parts, is that peculiar property of the straight line, which enters into every consideration of angles and parallels; and the neglect of which has been the cause of most of the embarrassment that has been felt in discussing the doctrine of parallel lines.

The definition of an angle here adopted, is that which was given by Euclid. It was also adopted by Clairaut and Laplace; and seems not to be susceptible of improvement. The inclination of the two lines is precisely the property which is used in all discussions in which an angle is an element.

The above definition of parallel lines is adopted because it is believed to characterize the relation of the parallels to each other, more precisely than those definitions which make the parallelism of the lines consist in their *not meeting* or their *being throughout at the same distance from each other*. Parallel lines are throughout at the same distance from each other, and cannot meet. These are truths which result from the property or rather the *relation* of parallelism; that is, from their having the same direction. This *identity of direction* is what constitutes the parallelism of the lines. And this notion of parallels should be, at first, presented to the contemplation of the learner; for this is the simple principle from which result all those propositions that make up what is called the doctrine of parallel lines.

The fundamental truth from which are deduced the various propositions respecting parallels, whatever may be the form in which it is stated, is essentially this:—*Parallel lines are equally inclined to a straight line meeting them.*

It seems to be pretty generally acknowledged that this proposition has never yet been geometrically proved, in any of the elementary treatises of geometry, though it is a fun-

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direction in all their parts, must have the same inclination to both. That is, *When a straight line meets two parallel straight lines, the angles which it makes with the one are equal to those which it makes with the other.* Clearer evidence of the truth of this proposition cannot be desired.

For those propositions which are generally proved by the method of *indivisibles*, or the *reductio ad absurdum*, or both, the method here employed is essentially the method of *limits* or *ultimate ratios*, used by Newton in his *Principia*, "to avoid," as he says, "the tediousness of deducing perplexed demonstrations *ad absurdum.*" Carnot says of the method of limits, that "it is of very great importance, as it relieves us from the necessity of using the *reductio ad absurdum*, the most *troublesome* of those operations which constitute the method of exhaustion." It may be added that, after the student has gone through the labor of committing perfectly to memory one of these troublesome arguments, there is frequently reason to doubt whether he comprehends the entire force of the process.

A second reason for preferring the method of limits to the argument *ad absurdum*, is, that throughout the whole of the process, it approaches directly the truth sought; and is substantially that process of the mind by which the truth may be supposed to have been discovered.

A third reason for adopting this method in preference to the others will be given in the words of Laplace. "The method of limits is the basis of the infinitesimal calculus. To facilitate the student's understanding of this calculus, it is useful to point out its earliest germs in elementary truths, which should always be demonstrated by methods the most general. The student thus gains, at the same time, knowledge and the means of increasing it. In his subsequent studies he merely follows the path which has been traced for him, and in which he has become accustomed to walk;

and thus his advancement in it will grow much less difficult. Moreover a system of knowledge is best preserved and extended by connecting its parts by a uniform method. In teaching, therefore, prefer the most general methods, endeavour to present them in the most simple manner, and you will find, at the same time, that they are almost always the most easy."

In discussing those subjects in which the three dimensions of space are concerned, the terms *solid* and *solid angle* are not used. The term *solid* is calculated to mislead the learner; as, in its common signification, it expresses a property of which geometry takes no cognizance.

Instead of the term *solid angle,* the recent method of designating the different angles formed by any number of planes, is here adopted. This nomenclature is useful, not only on account of its definiteness, as presenting to the mind distinct and precise objects of contemplation, instead of the more general notion; but also as an introduction to modern treatises on geometry of three dimensions.

It may be proper to remark, that in the whole course of the Elements, the author has left something for the learner *to do.* He has asked questions which he has left unanswered, proposed problems to be solved and truths to be proved, as an exercise of the learner's ingenuity; and has stated certain things as evident, which may require a moment's reflection from the young reader: but by this it is believed, his progress will be rather facilitated than impeded, as it secures his attention, and gives him the habit of deducing truths from premises without the assistance of another.

"If I must apologize," says Bézout, "for neglecting the use of the words *Axiom*, *Theorem*, *Lemma*, *Corollary*, *Scho-*

*lium*, &c., two reasons have determined me; the first is, that the use of these words adds nothing to the clearness of a demonstration; the second is, that this apparatus of terms may frequently divert the attention of beginners from the truth in question, by leading them to suppose that a proposition invested with the name of *Theorem* must be a proposition as remote from their knowledge, as the name is from terms with which they are familiar. But, to the end that those of my readers who shall open other books of Geometry may not imagine that they have fallen upon an unknown region, I think it proper to inform them that,

"*Axiom* signifies a self-evident proposition.

"*Theorem*, a proposition which makes a part of the science in question, but whose truth, in order to be perceived, requires a course of reasoning called a *demonstration*.

"*Lemma* is a proposition which is not necessarily a part of the theory in question, but which serves to facilitate the transition from one proposition to another.

"*Corollary* is a consequence which is drawn from a proposition that has just been established.

"*Scholium* is a remark upon what precedes, or a recapitulation of what precedes."

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The latter part of the volume contains an Introduction to Descriptive Geometry, a science whose practical connexion with so many liberal pursuits gives it a claim to be reckoned one of the regular branches of academical instruction; aside from the peculiar kind of discipline which it affords to the mind, the interesting nature of many of the topics which it discusses, and their effect upon the taste of the student.

By the study of Descriptive Geometry, the mind sees bodies and their parts in all their relations of position, mag-

nitude, and figure; it becomes accustomed to the contemplation of forms, and acquires a certainty and readiness of the imagination which enables it to make with variety and skill, new combinations of the elements of form.

Nor is Descriptive Geometry confined to the mere representation of visible forms; many of the severer subjects belonging to mathematical science, are thoroughly discussed by processes which this science has taught. "In a word, with the aid of a small number of the elementary propositions of geometry, this science possesses an almost infinite variety of means, by which we may arrive at the solution of the most difficult problems. It requires but a few weeks' study to be sufficiently understood; it advantageously replaces the common modes of practice, the long and laborious study of which is rendered unnecessary; at the same time, it gives us the immense advantage of treating with equal ease new combinations or unforseen cases." *

In offering to the public this Introduction to Descriptive Geometry, the Author makes no pretensions to originality. His object has been to comprise, within convenient limits, the fundamental principles of the science first given to the world by Monge, with a few examples of its application to linear perspective and some other important projections. It is, however, but an introduction; and though it gives the general principles of construction, is not meant to be considered as a *treatise*. If it should serve the purpose of calling the attention of teachers and the guardians of our academical institutions, to the importance of the subject, the object of this part of the work will have been obtained.

In preparing the Introduction to Descriptive Geometry, the writings of Monge, Hachette, and Lacroix have been

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* Crozet.

consulted, and also the treatises upon this subject of Cloquet and Professor Davies of West Point. To the latter treatise the reader is referred for a fuller discussion of Tangent and Cutting Planes and Spherical Projections; also for a discussion of Spherical Trigonometry and the subject of Warped Surfaces.

In Cloquet's treatise Descriptive Geometry is applied to the projection of a great variety of bodies, to the laws of Optics, to the determination of shadows, and to the science of perspective drawing. In the present treatise, the chapter upon Perspective is mostly taken from Lacroix.

There will be found, both in the *Elements* and in the Introduction to Descriptive Geometry, errors and slight omissions which can hardly be avoided in a first impression of a work of this kind; but none, it is hoped, which essentially affect the work.

*Cambridge, October* 28, 1829.

# CONTENTS.

The learner is requested, before he reads the book, to correct th
lowing Errata.

| | | |
|---|---|---|
| Page 3, | line 32, | after " *AEC* " insert " , the letter at the vertex placed in the middle." |
| 3, | 35, | for " as " read " is " |
| 5, | 29, | " angles." read " angles," |
| 17, | 43, | " if " read " since " |
| 21, | 33, | " sometime " read " sometimes " |
| 38, | 24, | " ACD " read " ACB " |
| 60, | 14, | " is is " read " is " |
| 72, | 29, 30, | " dimension " read " dimensions " |
| 77, | 26, | " FH " read " FGH " |
| 90, | 21, | " parallelopi-" read " parallelopip-" |
| 91, | 11, | " *z* " read " &c." |
| 92, | 31, | " F " read " E " |
| 94, | 2, | " *their ratio of the* " read " *the ratio of their* " |
| 135, | 28, | " MG′M′ " read " M″G′M′ " |
| 138, | 19, | " EF′ " read " EE′ " |
| | 24, | " ( *OM*," read " ( *OM*″," |
| 139, | 20, | " *AfE* " read " *AfF* " |
| | 22. | " developement " read " revolution " |
| | 29, | " F " read " *F* " |
| | 33, | " *Af* " read " *Af*′ " |
| 164, | 24, | " under " read " in " |
| 166, | 29, | " 105 " read " 106 " |

# ELEMENTS OF GEOMETRY.

## General Notions of Extension.

1. Geometry is that science which teaches us to investigate the magnitudes and forms of *extended things*, or *extended space*, and the relations of their parts.

2. *A body*, or the space which it occupies, is extended in *three directions;* it has *length*, *breadth*, and *thickness* or *depth*. If it were destitute of either of these *dimensions*, it would cease to be.

3. The space which a body occupies, is separated from other space by what we call the *surface* or the outside of the body. *A surface is extended in two directions;* it has length and breadth, but is destitute of thickness, and therefore makes no part of the body itself. A body then, and even definite space, is bounded by surface.

4. If the body have several *faces*, like a square block of wood, for instance, these faces may be considered as so many distinct surfaces, each of which is bounded by the *edges* formed by the meeting of this face with the other faces of the body. These limits are no part of the surface; they have neither breadth nor thickness; *they have only length*, and are called *lines*. The limit of a surface, therefore, is a line.

5. The line itself is limited by *a point;* which has *no extension*. A point may also be taken in a line, a surface, a body, or in extended space; it, however, makes no part of either of these magnitudes. It has *position*, but *no extent*. It may, by moving in space, be considered as *generating a line*.

## PART FIRST.

### SECTION I.—*The properties of straight lines and circular lines.*

6. Between two assumed points (as A and B in
Fig. 1. *fig.* 1), several lines may be drawn. Among these there may be one which tends the same way in all its parts; this is the *straight line.* We therefore define *a straight line* to be, one which *has the same direction throughout its whole extent.* It is evident, that *the straight line is the shortest which can be drawn between two points.* It is also evident, that *between two points only one straight line can be drawn.*

7. A line which continually changes its direction is called a *curve* line.

Curve lines admit of great variety in the character of their curvature, whereas among straight lines, the only variety is that of *length.*

Of the various curves, the only one whose properties are made the subject of *elementary* geometry, is *the circular curve.* This is the most simple of all curves, its curvature being the same in every part. It is described by one extremity of a straight line, revolving about the
Fig. 2. other extremity which remains fixed. In *fig.* 2, if the straight line AC be made to revolve about the point C which does not change its position, the point A will describe the circular curve ABDA. This line, in reference to the *circle* ABDA, is called its *circumference;* and it is manifest that every point in this curve is equally distant from the point C, called *the centre of the circle.* The lines AC, BC, DC, &c, which measure this distance, are called *radii* of the circle. A straight line passing through the centre and terminating in the circumference, is called *a diameter.* Any portion of this curve, as BD, is called *an arc of the circle.*

In the same circle are the radii equal? and why?

How does the diameter compare with the radius?

8. A surface to which a straight line can be *applied* in every direction, so as to touch the surface through the whole extent of the line, is called *a plane surface,* or simply a *plane.* A surface which is neither a plane surface nor composed of planes, is called a *curve surface.* Such is the surface of a ball, a roll of paper, &c.

9. *Two straight lines which have the same direction in space*, are called *parallel lines* (*fig.* 3). As they have the same direction, they can neither approach towards, nor diverge from each other; *parallel lines*, therefore, *must be throughout at the same distance from each other; and, however far produced, can never meet.* Fig. 3.

10. Where two straight lines taken in the same plane are not parallel, they must incline the one to the other, and are said to make an angle with each other (*fig.* 4). *An angle*, then, *is the inclination of one line to another.* The *magnitude* of the angle depends upon the degree of inclination, and not at all upon the distance or length of the lines. Fig. 4.

11. The nearer two lines are to being parallel, the smaller the angle, which becomes nothing when the lines become parallel. It increases as the lines diverge; and when this inclination is the same on both sides (as AB and CD, *fig.* 5), the angles are called *right angles.* In this case, the line AB is said to be *perpendicular* to the line CD. CD is also perpendicular to AB. Fig. 5.

When two lines are neither parallel nor perpendicular to each other, they are said to be *oblique* to each other.

*Remark.* We have, then, in geometry, *four kinds* of magnitudes. 1. That of a line. 2. That of a surface. 3. That of a body. 4. That of an (inclination or) angle. Angles may be *added*, *subtracted*, *multiplied*, and *divided*, like other magnitudes or quantities.

12. Two lines which form an angle in a plane, either meet, or being produced, will meet each other in some point, the point E, for instance, as in *fig.* 6. This point of meeting is called *the vertex of the angle.* We *express the angle*, usually, by three letters, thus, *the angle AEC*, When there is only one angle at the same vertex, we express it by a single letter; as *the angle* E (*fig.* 7), that is, the angle whose vertex is at E. Fig. 6. Fig. 7.

13. Suppose the line AB perpendicular to CD (*fig.* 8) to move about the point B, into the position A′B, the angle A′BC will be greater than a right angle, and the angle A′BD, will be less than a right angle; but the amount of angular space is not changed by this movement; the angle ABA′ is taken from one of the angles and added to the other; the *sum* of the two angles is the same as before. Hence we say, that, *when one straight line meets another straight line, the sum of the two angles is equal to two right angles.* Fig. 8.

Fig. 8. The sum of the angles being equal to two right angles, the exterior lines CB, BD, must be in *the same straight line;* otherwise something must be added to or subtracted from the angular space to make CBD a straight line; which would make the sum of the two angles greater or less than two right angles.

*Remark.* An angle which is less than a right angle is called an *acute* angle; as A′BD. An angle greater than a right angle is called an *obtuse angle*, as the angle A′BC. Both obtuse and acute angles are called *oblique angles.*

14. If other lines were drawn to the same point B, on
Fig. 9. the same side of CD, as in figure 9; the whole amount of angular space, would evidently be neither increased nor diminished by it; but would still be equal to two right angles. And we hence derive this general truth, that,—*When a straight line is met by several other straight lines, in the same point, the sum of all the angles on the same side of it, is equal to two right angles.*

15. If all these lines be produced through B, it is manifest that the angular space *below* the line CD, will be equal to the angular space *above;* therefore,—*The sum of all the angles made by several straight lines diverging from the same point, is equal to four right angles.*

16. As a straight line has the same direction in all its parts (6), two straight lines must have the same inclina-
Fig. 10. tion in every part (*fig.* 10); therefore if they intersect each other, *the angles which are opposite, at the vertex,* (called vertical angles,) *are equal.* Each of the two angles AEC and DEB, expresses the inclination of the two straight lines AB and CD, to each other; they are therefore equal angles.

If the two lines are perpendicular to each other, the two angles on the same side of either of them, are equal; but these are respectively equal to their *vertical angles* (16); it is therefore evident that *all right angles are equal.*

## *Of Parallel Lines.*

17. The properties of parallel lines are deduced from this fundamental proposition. *Two straight lines which*

*have the same direction in space, must be equally inclined to the same straight line which meets them.*

This proposition results directly from the definition of *a straight line.* If two straight lines have different inclinations to the same straight line, that is, to the *same direction*, they must be inclined to each other, and therefore cannot be parallel.

18. Suppose the straight line AB (*fig.* 11) to intersect the two parallel straight lines CD and FG; it must have the same inclination to both. This inclination is expressed by either of the angles AEC, AHF; they are therefore equal. As these angles have correspondent situations in relation to these two intersections, they are called *corresponding angles.** And as nothing in this analysis depends upon any particular inclination of AB to the parallels, we derive this general truth: *When a straight line intersects parallel straight lines, the corresponding angles are equal.* Fig. 11.

19. The sum of the two angles AEC and CEH, is equal to two right angles (13); if instead of the angle AEC, we substitute its equal EHF (18), we shall have the sum of the angles CEH and EHF, equal to two right angles. These angles are called *interior on the same side of the cutting line;* and we say that—*When a straight line cuts two parallels, the sum of the two interior angles, on the same side, is equal to two right-angles.* And as the sum of the four angles on the same side of the cutting line, or *secant*, must be equal to four right angles. *The sum of the exterior angles on the same side, is equal to two right angles.*

20. The angle EHF is equal to its corresponding angle AEC (18), and the angle DEH is equal to the angle AEC, being *vertical* with it (16). Therefore the two angles, DEH and EHF, (each equal to AEC,) are equal to one another. These angles are *internal* with respect to the parallels, but on *alternate sides* of the cutting line. Whence we derive the general truth,—*A straight line cutting parallels, makes the alternate-internal angles equal.*

21. The angle GHB is equal to the angle DEH (18), and therefore equal to AEC, vertical with DEH. These

* They are sometimes called *internal-external angles.*

Fig. 11. two angles AEC and GHB, are *exterior* with respect to the parallels, and on *alternate sides of the secant;* they are therefore called *alternate-external;* and we have this general proposition,—*When a straight line intersects two parallel straight lines, the alternate-external angles are equal.*

22. These five propositions may be summarily enunciated as follows: When a straight line meets two parallel straight lines,

(1.) The corresponding angles are equal.

(2.) The sum of the two interior angles upon the same side, is equal to two right-angles.

(3.) The sum of the two exterior angles on the same side is equal to two right-angles.

(4.) The alternate-internal angles are equal to each other; and

(5.) The alternate-external angles are equal to each other.

23. If we incline one of the parallels in question, FG for instance, the other two lines remaining the same, it will change the magnitude of each of the angles at H; and neither of the five propositions will be true. We have therefore the converse of these propositions; and we say that, *when a straight line meets two straight lines, and the corresponding angles are equal; or the sum of the two interior angles on the same side, or the sum of the two exterior, is equal to two right angles; or the alternate-internal angles equal to each other, or the alternate-external angles equal to each other, the two lines must be parallel.*

*Cor.* If the two interior angles are *less* than two right-angles, the lines incline towards each other, and will meet on that side, if produced sufficiently far. If the sum of the interior angles on one side is *greater* than two right angles, the sum of those on the other side, is less than two right-angles (13), and therefore the lines will meet on that side, if produced sufficiently far.

Fig. 12. 24. Let the two angles ABC, DEF, (*fig.* 12) have the side AB, parallel to DE, and BC parallel to EF; produce BC to I, and also produce DE till it meet BI in H. Then on account of the parallels EF and BI, the corresponding angles DEF and DHI, are equal (22); and with reference to the parallels DH and AB, the corresponding angles DHI and ABC are equal; that is, ABC

is equal to DEF; and universally—*Two angles which have their sides parallel and directed the same way, are equal.*

25. To measure a straight line, we apply to it a *scale* or rule, or *some other straight line* of known and standard measure, and thus ascertain its length. This corresponds with general practice in analogous cases; one quantity is usually measured by another quantity of the same kind.

In measuring an angle, geometers have adopted a method somewhat different. To ascertain the magnitude of an angle, they measure the portion of a circular arc embraced by the two sides of the angle, the vertex of the angle being at the centre of the circle.

26. To obtain a clear idea of the magnitude of an angle, and the connexion which it has with a circular arc, let us suppose (*fig.* 13) that at first the two lines AC and BD, coincide in the part BC, and that the part A, of the line AC, be raised, so that this line, departing from BC, may revolve about the point C; it becomes immediately inclined to BC; and this inclination increases as the arc described by the point *a* increases, and in the same degree; that is, for the same amount of angular motion in any part of the revolution of the line AC, the arc described by the point *a* will be the same; so that when two angles are equal, the arcs drawn with the same radius, from the vertices of the angles, as centres, will be also equal. We hence see how the magnitude of an angle may be designated by a circular arc. Fig. 13.

For this purpose, the ancients divided the circumference of the circle into 360 equal parts, called degrees; each degree into 60 minutes; and each minute into 60 seconds. And the magnitude of an angle they expressed by the degrees, ($^\circ$) minutes ($'$) and seconds ($''$,) which express the *value*, or magnitude, of the arc comprehended between the sides, the vertex being at the centre of the circle; thus, an angle of 35 degrees, 27 minutes, 15 seconds; usually written $35^\circ\ 27'\ 15''$.

27. We have seen (15) that the sum of all the plane angles made at the same point, is equal to *four* right angles; and it is manifest that the sum of all the arcs, which they would embrace in a circle described from their common vertex as a centre, would be the entire circumference. A circumference then, or $360^\circ$, is the

measure of *four right angles.* If, therefore, through the
Fig. 14. centre of the circle (*fig.* 14) we draw two diameters, perpendicular to each other, we shall divide the circle into four equal parts, called *quadrants.* ABC is a quadrant; the arc AB, is an arc of 90°, and the measure of the angle ACB, which is a right-angle. *A right angle then is an angle of* 90 *degrees.* If we divide the arc AB into 90 equal parts, and from C through the 55th division, draw the line CD, we shall have the angle ACD, an angle of 55°; and the angle BCD, an angle of 35°. The obtuse angle ACD′ is measured by the arc AD′, greater than a quadrant.

28. Of the two acute angles DCA and DCB, each is called *the complement* of the other, because each is just what the other wants to make it a right-angle. The acute angle DCA, and the obtuse angle DCA′, are called *supplements* of each other, because each being subtracted from the sum of two right-angles, or 180°, will give the other. Angles *complemental* to each other, are *both acute.* Of two angles *supplemental* to each other, if one be acute the other will be obtuse, and *vice versâ.* Two right-angles are supplements of each other.

29. A method practised by surveyors for measuring angles in the field, is by means of an instrument called
Fig. 15. *a semicircle*, (*fig.* 15). At the two extremities of the diameter AB, *sights* are fixed, through which you look directly along the diameter. To the centre C, a moveable *index* is attached, with sights at its extremities, D, E, so that the line DCE is a straight line. When an angle of a field is to be measured, this instrument is placed *horizontally* with its centre C at the vertex of the angle, or, in other words, exactly at the corner of the field. The instrument is so placed that, by looking through the sights A, B, we look along one side of the field; the index DE is then turned about the central point C, till the line of sight is directly along the other side of the field. The magnitude of the angle will be expressed by the number of degrees, minutes, &c. in the graduated arc BE, comprehended between the stationary and moveable diameters.

30. When these angles are to be transferred to paper, for the purpose of giving a *plan* of the ground; an instrument called a *protractor*, is used. This is usually a
Fig. 16. semicircular piece of brass (*fig.* 16), graduated in de-

grees and parts; and the centre C accurately determined. When an angle is to be set off upon the plan, one of the lines embracing the angle is first drawn; the protractor is then placed with its centre at the point where the vertex of the angle is to be, and its semidiameter lying along this line. The number of degrees, &c, is then counted off upon the arc, or *limb* (as it is called) of the instrument, and the point carefully indicated; then through the centre and this point a line is drawn, and we have the angle required.

31. When the magnitude of an angle already constructed, is required, the protractor is placed with its centre at the vertex of the angle, and its diameter lying along one of the sides; the number of degrees &c. will be read upon the graduated limb, at the point where it is cut by the other side of the angle.

32. *To construct an angle equal to a given angle without the protractor.* Suppose ACB (*fig.* 17) to be Fig. 17. the given angle; and suppose at *c*, in the straight line *cb*, it is required to make the angle *acb*, equal to ACB. With a convenient radius CD, from C as a centre, describe the arc DE; and from *c*, as a centre with the same radius, describe the arc *d e*; then from *d* as a centre with a radius equal to DE, describe another arc cutting the arc *d e*, in the point *e;* and through the points *c* and *e*, draw the line *c e a*, and you have the angle *a c b*, equal to the angle ACB.

It is evident that, in the angle ACB, if the sides were more inclined, that is, if the angle were increased, the distance of the points D and E would be increased; and if the angle were diminished this distance would be diminished; therefore, while the distance of these points is the same, and their distance from the vertex of the angle is not changed, the angle must be the same; but these conditions are the same in the two angles ACB, *acb;* they are therefore equal.

## *Of Plane Figures.*

33. A *plane* has been defined,—a surface, to which a straight line, being applied in every direction, will touch the surface in its whole extent. *Plane figures* are portions of plane surface bounded or enclosed by lines.

Those bounded by straight lines, or *right lines* (as they are frequently called), are denominated *rectilinear figures.* Those bounded by curve lines are *curvilinear figures.*

34. Among rectilinear figures, as two straight lines cannot enclose a space, the simplest is that of *three sides,* called the *triangle.* Triangles are differently denominated, according to the different relations of their parts. When *the three sides are of equal length* it is called an
Fig. 18. *equilateral triangle* (*fig.* 18). When *two of its sides are equal,* it is called an *isosceles triangle.* When *no two sides are equal,* the triangle is called *scalene*
Fig. 19. (*fig.* 19). When it has *one right angle,* it is called *a*
Fig. 20. *right-angled triangle* (*fig.* 20). In the right-angled triangle, the side opposite the right-angle is called the *hypothenuse.*

35. The first inquiry concerning the properties of triangles, respects *the sum of the three angles; is it always*
Fig. 21. *the same?* Let us take the triangle ABC (*fig.* 21), and through the vertex A draw the straight line DE parallel to the *base* BC; we then have AB, a straight line meeting the two parallel lines DE and BC; the *alternate-internal* angles DAB and ABC, are equal; and AC being a straight line meeting the same parallels, the angles EAC and ACB, are equal, for the same reason; then the angle DAB being equal to the angle B, and the angle EAC, equal to the other angle C, the three angles DAB, BAC and CAE, are equal to the three angles of the triangle; but the sum of these is equal to two right-angles (14); therefore, the sum of the three angles of the triangle is equal to two right-angles. It is evident that, however the sides and angles of the triangle may be changed, they will always admit of a straight line (as DE) being drawn through the vertex of one of the angles, and parallel to the opposite side; and that the two outer angles at this vertex must therefore be equal to the other angles of the triangle. We say, therefore, that, *the sum of the three angles of every triangle, is equal to two right-angles, or* 180°.

36. (1.) If we know two angles of a triangle, how can we find the remaining angle?

(2.) How many obtuse angles can a triangle have? Why?

(3.) How many right angles can any triangle have? Explain.

(4.) If one angle of any triangle, be a right-angle, what will be the sum of the other two? Fig. 21.

(5.) If one of the oblique angles of a right-angled triangle be given, how can we find the other?

37. If we produce the base of the triangle BC to F, the angle which it makes with AC on the outside is called the *exterior angle;* and because AC is a straight line meeting the two parallels DE and BF, this angle is equal to the angle DAC; but DAC is composed of the two angles DAB and BAC; and DAB is equal to the angle B; therefore this exterior angle ACF, is equal to the sum of the two angles ABC and BAC, of the triangle; these two angles with respect to the exterior angle, are called *interior and opposite.* We say then—*The exterior angle, made by producing one of the sides of the triangle, is equal to the sum of the two interior and opposite angles.*

38. *Problem.* The three sides of a triangle being given, to construct the triangle.

Let the three given sides be the lines A, B, C, (*fig.* 22). Fig. 22.
Draw the line DE equal to the given line A; then from D as a centre with a radius equal to the given line B, describe an arc; and from E as a centre, with a radius equal to the other given line C, describe an arc cutting the other arc in F; draw DF and EF, and you have the triangle required.

39. It is plain that no different triangle can be formed with these three lines. The only different construction which the case admits, is to make the triangle on the lower side of the base, as the triangle D′E′F′; but this triangle is not really *different* from the first. To show this, turn the last triangle over by lifting up the part F′ and making the whole turn about the base D′E′; then place it upon the first so that the point D′ will be upon D, and E′ upon E; this may be done, as each of the bases is equal to the given line A. The triangles will then coincide in all their parts, and must therefore be equal. The point D′ being upon D, the point F′ must be at the same distance from D, as the point F is, it must therefore be in the first described arc; and as E′ coincides with E, F′ must be at the same distance from E, that F is; it must therefore be somewhere in the other arc which crosses the first in F; if F′ is in each of these two arcs, it can only be at their intersection, and there-

Fig. 22. fore falls upon the point F, and the two triangles coincide in all their parts; they are not different, therefore, but *equal;* and we say, universally, *When two triangles have the three sides of the one equal to the three sides of the other respectively, the angles will also be equal, respectively, and the two triangles will be equal in all their parts.* This is the *first case of equal triangles.*

40. *Problem.* Two sides of a triangle being given, and the angle contained by these sides, to construct the triangle.

Fig. 23. Let A and B (*fig.* 23) be the two given sides, and C the given angle. Draw DE equal to the given line A; at D make the angle EDG, equal to the give angle C, (32) and produce the side DG to F, making DF equal to the given line B; then join FE and you have the triangle required.

41. It is evident that no different triangle can be constructed with these two sides containing an angle equal to C. The only different construction with these things given, would be to make the triangle on the other side of the base DE; but it is plain that the triangle thus formed would only be the triangle DEF turned over; the two triangles would therefore be equal in all their parts. And we say universally,—*When two triangles have two sides of the one equal to two sides of the other, each to each, and the angle contained by these two sides of the one, equal to the angle contained by the two sides of the other, the two triangles are equal in all their parts.* This is the *second case of equal triangles.*

42. *Problem.* One side of a triangle, and the two angles adjacent to that side, being given, to construct the triangle.

Fig. 24. Let the line A (*fig.* 24) be the given side, and the angles B and C, be the angles adjacent to that side. Draw the line DE equal to A, for the base of the triangle; then at the point D, make an angle equal to the given angle B; and at the point E, make another angle on the same side of DE, equal to the other given angle C; produce these two sides till they meet in F, and the triangle is constructed.

43. It is manifest that no different triangle can be formed with these three things given. We may, as in the other cases, construct the triangle on the lower side of the base DE; but the triangle thus formed might be

turned over, and placed upon the triangle DEF, and because the bases are equal, and the angles at the bases the same, the side DF′ would fall upon DF, and the side EF′ would fall upon EF; the point F′ would therefore fall directly upon the point F; and the two triangles would entirely coincide in all their parts. We say, therefore—*If two triangles have a side and the two adjacent angles of the one, equal to a side and the two adjacent angles of the other, each to each, the two triangles will be equal in all their parts;* that is, the third angle of the one will be equal to the third angle of the other, and the two remaining sides of the one will be equal respectively to the two remaining sides of the other. This is the *third case of equal triangles.*

44. *Problem.* The hypothenuse and one other side of a *right-angled* triangle being given, to construct the triangle.

Let the line A (*fig.* 25) equal the hypothenuse, and the line B, the other given side of the triangle. Draw the line CD equal to the given side B: and at D draw the indefinite line DE, perpendicular to CD; then from C as a centre, with a radius equal to the given hypothenuse, draw an arc cutting the perpendicular in E; draw CE, and you have the right-angled triangle required. Fig. 25.

[Let the learner show that no different triangle could be constructed with these things given.]

We say then—*If two right-angled triangles have the hypothenuse and a side of the one, equal respectively to the hypothenuse and a side of the other, the two triangles will be equal in all their parts.*

This is the *fourth case of equal triangles.*

45. We have seen that the *sum* of the three angles of every triangle, is equal to two right-angles; yet there is an infinite variety in the magnitude of the individual angles; and this evidently depends upon the relative magnitudes of the sides. We have learned that when the sides are the same, the angles must be the same (39). Let us examine this subject a little farther. We will take the isosceles triangle ABC (*fig.* 26), of which AB and AC are the two equal sides. From the vertex A, to the middle of the base BC, draw the straight line AD. This will divide the triangle ABC, into two triangles ADB, ADC. If we examine them, we shall see that AB of the one is equal to AC of the other, that DB Fig. 26.

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Fig. 26. of the one is equal to DC of the other, and that AD is a side common to both; they are therefore equal by the *first case of equal triangles;* and if ABD were turned over upon ADC, we should find that the two would coincide in all their parts. The angle B is therefore equal to the angle C; the two angles whose vertices are at A, equal to each other; and the two angles whose vertices are at D, equal to each other; the two last are right angles of course (11). Whence we derive the following important truths—1. *In an isosceles, triangle, the angles opposite to the equal sides, are equal.* 2. *A straight line drawn from the summit of the isosceles triangle to the middle of the base, is perpendicular to the base, and bisects the angle whose vertex is at the summit.*

Fig. 27. 46. If the triangle had been *equilateral* (*fig.* 27), that is, had had its three sides all equal, either side might have been taken for the base. This would have proved the angle A equal to the angle B, and the angle C equal to the angle B; hence—*An equilateral triangle is also equiangular;* that is—*has all its angles equal.*

As the sum of the three angles of every triangle, is equal to two right-angles, or 180 degrees; *in an equilateral triangle, each angle is an angle of 60 degrees.*

If you have the angle whose vertex is at the summit of an isosceles triangle, how can you find the other angles?

If you have an angle at the base of an isosceles triangle, how will you find the other angles?

Fig. 28. 47. Take next the *scalene* triangle ABC (*fig.* 28). Let AB be the greater side; take upon AB, the part AD equal to AC, and draw DC. The angle ADC may be considered as an exterior angle to the triangle DBC, and therefore greater than the angle B; the triangle ADC is isosceles, and consequently has the angle ACD equal to the angle ADC; but the angle ACB is greater than ACD; for a still stronger reason is the angle ACB, greater than ABC. But the side AB is greater than AD, or than its equal AC. We see then that the greater side is opposite to the greater angle. If we had compared either of these angles with the angle A, and also their opposite sides, the result would have been similar; whence we have this general rule—*In any triangle, the greater side is opposite to the greater angle.* The side opposite to an angle is frequently said to *subtend the angle.*

48. If two sides of a triangle are unequal, the opposite angles will be unequal (47); therefore—*If in any triangle two of the angles are equal, the sides opposite those angles will be equal, and the triangle will be isosceles.*

49. In a right-angled triangle, as the two oblique angles must be each less than a right-angle,—*the hypothenuse* (the side opposite the right angle) *must be greater than either of the other sides.* The two sides which contain the right-angle are frequently called *the legs of the triangle.*

50. *Either side of a triangle,* being a straight line and therefore the shortest which can be drawn between the vertices at its extremities, *must be less than the sum of the other two sides.* It follows from this, that *either side of a triangle must be greater than the difference between the other two sides;* if this were not the case, the greatest would exceed or at least equal the sum of the two others.

51. Suppose AB (*fig.* 29) perpendicular to DC, and AE and AF two lines drawn from the same point A, and falling obliquely upon DC at equal distances from the point B; the two triangles ABE and ABF, will have AB a common side, EB of the one equal to BF of the other, and the angle ABE equal to the angle ABF (11); the triangles will be equal by the second case, and will have the hypothenuses AE, AF equal, and each greater than the perpendicular AB (49). Draw AG at a greater distance from the perpendicular than AE is; you will have the triangle AEG obtuse-angled at E, because this angle is the supplement of the acute angle AEB (28); and in the triangle AEG, the side AG being opposite to the obtuse angle is greater than AE opposite an acute angle; and we have these three propositions. *If from a point without a straight line, a perpendicular be drawn to that line, and also several oblique lines,*— Fig. 29.

(1.) *The perpendicular is less than either of the oblique lines.*

(2.) *Oblique lines equally distant from the perpendicular are equal; and*

(3.) *Of two oblique lines, that is the greater which is at the greater distance from the perpendicular.*

52. It follows from the second of these propositions, that—*If a perpendicular be drawn through the middle of a line, AB, every point in the perpendicular will be equally distant from the two extremities of this line;* for

Fig. 29. straight lines drawn from any point in the perpendicular to the two extremities, will be oblique lines drawn at equal distances from the perpendicular, and therefore equal.

53. As two points are sufficient to fix the position of a straight line; *If one straight line pass through the middle of another straight line* [as AB through the middle of EF]; *and if any other point in the first of these straight lines, be at equal distances from the two extremities of the second, these lines will be perpendicular to each other.* And for the same reason—*If each of two points in a straight line, be equally distant from the two extremities of another straight line, the first will bisect the second, and be perpendicular to it.* To *bisect* is to divide into two equal parts.

54. *Problem.* From any point in a given straight line, to draw a perpendicular to that line.

Fig. 30. Let A be a given point in the line BC (*fig.* 30), from which it is proposed to draw a perpendicular. Take two points B and C in the given line, equally distant from the given point A; from each of these points B and C, with any radius greater than AB, draw arcs cutting each other in some point D, above or below the given line; and from A, draw a line through D, which will be the perpendicular required (53).

It is evident that there can be only one line perpendicular to BC, and passing through the point A; for if there could be two, as they make the same angle with the line BC, they will both have the same direction; and having the same direction, and passing through the same point, they must coincide entirely, and be one and the same straight line. *There cannot, therefore, be more than one perpendicular drawn through the same point in the same straight line.*

55. *Problem.* From a given point *without* a straight line, to draw a perpendicular to that line.

Fig. 31. Let A (*fig.* 31) be the given point, and BC the given straight line. From A as a centre, with any radius greater than the distance of the point A, from the given line, draw an arc cutting the given line in the points D and E; then from each of these two points D and E, with a radius greater than half of DE, draw arcs cutting each other in F; and from the point A, in the direction AF, draw the line AG; this line will be the per-

pendicular required; as it has two points each equally distant from the extremities of the line DE (53).

56. ***Problem.*** From a given point without a straight line, to draw a perpendicular to that line, when the perpendicular will come near one extremity of the line.

Let A (*fig.* 32) be the point from which it is required to draw the perpendicular to the line BD. From B as a centre with a radius equal to the distance BA, describe an arc below BD; and from E as a centre with a radius equal to the distance EA, describe an arc cutting the first arc in A′; then from A in the direction AA′, draw the line AG, which will be the perpendicular required. As B and E are two points in the straight line BG, each equally distant from A and A′; BG must be perpendicular to AA′ (53); that is, BGA is a right-angle. Fig. 32.

Any other line drawn through A, must be different from AG, and therefore differently inclined to BG; but AG is perpendicular to BG; any other line must consequently be oblique to BG. We say therefore—*From the same point without a straight line, only one perpendicular can be drawn to that line.*

57. ***Problem.*** To bisect an angle, as the angle C (*fig.* 33). Upon the sides, take CA and CB equal to each other; draw AB, and from C draw CD perpendicular to AB; CD will bisect the angle ACB (45). Fig. 33.

58. ***Problem.*** Through a given point, A (*fig.* 34), to draw a straight line parallel to a given straight line, BC. Fig. 34.

From A as a centre, with a radius sufficiently great, describe the arc ED; and from E as a centre with the same radius describe the arc AF; take EG equal to AF, and through A and G draw a straight line, which will be parallel to BC: because the alternate-internal angles GAE and AEF, being measured by equal arcs, are equal (26), and consequently the two lines AG and BC, to which they refer are parallel.

59. Suppose two parallel lines, AB and CD (*fig.* 35), to be cut by two other parallel lines, EF, GH, the space IKML, comprehended by these parallels, is called a parallelogram. We say then—*A parallelogram is a quadrilateral (or four-sided) figure of which the opposite sides are parallel.* The interior angles BIK and IKM are equal to two right angles (22); and the interior angles IKM and KML, are equal to two right angles; therefore, IKM added to either of the two angles BIK or KML, gives the Fig. 35.

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Fig. 35. same sum, BIK must be equal to KML. In the same manner we may show that the two angles IKM and ILM are equal to each other. We perceive that, in both cases, the equal angles are opposite to each other; we therefore say—*In every parallelogram, the opposite angles are equal.*

60. If we draw through the vertices K and L, the straight line KL, called a *diagonal* of the parallelogram; it will be a straight line meeting the two parallels AB and CD; the alternate angles ILK, LKM are equal (22); and because the straight line KL meets the two parallels EF and GH, the alternate angles IKL, KLM, are equal. We have then, the two triangles IKL, KLM, of which KL is a common side, and in which the angles IKL, KLI, of the one are respectively equal to the angles KLM, LKM, of the other; the triangles are therefore equal by the *third case* of equal triangles; and the sides opposite the equal angles will consequently be equal, viz. IL to KM, and IK to LM. But these equal sides are parallels comprehended between parallels. From which we derive these general truths:

(1.) *The diagonal of a parallelogram divides it into two equal triangles.*

(2.) *The opposite sides of a parallelogram are equal.*

(3.) *Parallel lines comprehended between parallel lines are equal.*

Fig. 36. 61. Suppose AB and CD (*fig.* 36) to be two parallel and equal lines. By joining their extremities by the straight lines AC and BD, we have the quadrilateral ABDC; and by drawing the diagonal CB, we have two triangles, ABC and CBD. Since AB and CD are parallel, and CB is a straight line meeting these parallels, the alternate angles ABC, BCD, are equal; and as AB and CD are equal, and CB a side common to the two triangles, these triangles are equal by the *second case*, and therefore equal in their corresponding parts; namely, the angle ACB is equal to the angle CBD; but these angles are alternate-internal with respect to the two lines AC and BD, and the secant BC; these lines are therefore parallel (23), and the figure ABDC is a parallelogram. We therefore say—*When, in any quadrilateral, two opposite sides are equal and parallel, the other two sides will be equal and parallel; and the figure will be a parallelogram.*

62. As two adjacent angles of a parallelogram are

equal to two right angles, the sum of all the angles is equal to four right angles. And if one of the angles of a parallelogram is a right angle, they will all be right angles; and the figure will be what is called *a rectangle.*

If two adjacent sides of a parallelogram are equal; as the sides opposite are equal to them, the sides will *all* be equal. In this case, if the angles are right-angles, the figure is called *a square.* If the sides of a parallelogram are *equal,* and the angles *oblique,* the figure is called *a rhombus.* If the sides are *unequal* and the angles *oblique* the figure is called *a rhomboid.*

63. If in the rhomboidal parallelogram ABDC (*fig.* 37) Fig. 37.
we draw the diagonals AD and CB, and compare the triangles AEB, CED, we shall have the two angles BAE, EDC, alternate-internal with reference to the parallels AB and CD; they are therefore equal; the two angles AEB and CED, are vertical angles (16), and therefore equal; consequently the third angle ABE, of the one, is equal to the third angle ECD, of the other (35); and the side AB is equal to the side CD, being opposite sides of the parallelogram. We have, therefore, the side AB, and the adjacent angles A and B, in the one triangle, equal respectively to the side CD and the adjacent angles D and C of the other; the triangles are consequently equal by the *third case;* and their corresponding parts are equal; namely, AE is equal to ED, and CE is equal to EB; that is, these diagonals bisect each other at E. It is evident that this fact results from the equality and parallelism of the opposite sides; whence we have the general truth—*In every parallelogram, the two diagonals bisect each other.*

64. If the parallelogram ABDC were a square or a rhombus, ABD would be an isosceles triangle (34), and BE would be perpendicular to AD (45). Hence—*In a square or in a rhombus, the two diagonals bisect each other at right angles.*

65. A *trapezoid* is a quadrilateral which has *two of its sides parallel,* and the other two oblique, to each other. If in a quadrilateral *no two of its sides are parallel,* the figure is called *a trapezium.*

*Remark.* Rectilinear figures of *five* sides are called *pentagons;* of *six, hexagons;* of *seven, heptagons;* of *eight, octagons,* &c. But all rectilinear figures are comprehended under the general name of *polygons.* The

sum of all the sides of a polygon is called its *perimeter.*

66. *Problems.* (1.) Having two contiguous sides and the contained angle, in a parallelogram, to construct the parallelogram.

Fig. 38. Let A and B (*fig.* 38) be the given sides, and C the given angle. Draw DE equal to A; and at D, make the angle EDF, equal to the given angle C; and make DF equal to B; then through F draw FG parallel to DE, and through E draw EG parallel to DF, and the parallelogram is completed.

(2.) Construct a square upon an assumed line, as one of its sides.

(3.) Construct a rhombus upon an assumed line, and having an assumed angle.

These examples are sufficient to enable us to construct any rectilinear figure, when we have the sides given, and the angles which they make with each other. In this way the *plan* of a town or an estate is constructed.

67. We have seen that the sum of all the angles in a parallelogram is equal to four right angles (62). It is desirable to know whether there is any general law respecting the angles of polygons.

Fig. 39. Let us take the polygon ABCDE (*fig.* 39); produce each of its sides outward in succession; and from some point in the interior of the polygon, draw lines parallel to each of the sides. It is evident that each of the angles, *a*, *b*, *c*, &c. must be equal to the corresponding *exterior* angles of the polygon (24); but the sum of all the angles made by lines diverging from the same point in a plane, is equal to four right angles (15). It is manifest that how many sides soever the figure may have, from the same point lines may be drawn respectively parallel to each of the sides. We say therefore—*The sum of all the exterior angles of every polygon, made by producing the sides outward in succession, is equal to four right angles.*

68. It will be perceived, that at each vertex of the polygon, the sum of the interior and exterior angles is equal to two right angles (13); consequently the sum of all the interior and exterior angles is equal to as many times two right angles as the polygon has sides. But the sum of all the exterior angles is equal to four right-angles (67); therefore—*The sum of all the interior angles of any convex polygon, is equal to as many times two right-angles as the figure has sides, wanting four right-angles.*

*Remark.* These two propositions are limited to *convex* polygons; or such as have all their angles *salient*, that is, having the vertices of the angles outward. In polygons which have *re-entering* angles, that is, angles whose vertices are towards the interior of the polygon, as the angle C (*fig.* 40), these propositions will not be true. Fig. 40.

69. It is evident that any polygon, ABCDEF (*fig.* 41), may be divided into triangles, by drawing from one of the vertices, as A, diagonals to each of the other vertices not adjacent. And it is manifest that if, in two equal polygons (that is, in two polygons which would coincide if placed the one upon the other, which is called *super-position)*, diagonals be drawn from corresponding vertices in the two polygons, to the opposite vertices, the triangles in the one polygon, would be equal respectively to the triangles in the other; and the triangles being equal, and similarly disposed, the polygons which they compose must be equal. That is—*Two equal polygons may be divided into the same number of triangles, respectively equal.* And—*If two polygons are composed of the same number of triangles, equal and similarly disposed, the polygons are equal.* Fig. 41.

70. A more important method of *determining* a polygon by triangles, is that in which the triangles have for a common base, one of the sides of the polygon, as AB (*fig.* 42), and their summits respectively in the opposite vertices of the polygon. It is manifest that the triangles ABC, ABD, ABE, ABF, having AB for a common base, determine the points C, D, E, F; all the vertices of the polygon are therefore determined, and consequently the polygon itself. Fig. 42.

71. In this way surveyors sometime determine the most important points in a field or an estate, whether on its border or in the interior. A straight line is measured, from each extremity of which all the points to be determined can be seen. This line is called the *base line*, because it is made the base of a series of triangles whose summits are to determine the exact position of the other points. No point can in this way be determined, which is in the base line produced. This method of determining the points in a survey, is called *triangulation*.

72. We have discussed the general properties of individual triangles, and their relative parts. We have also

compared togéther such as are equal; that is, such as may be placed, the one upon the other, so as to *coincide entirely*. And we remark that this *coincidence* is an indispensable condition of *equal figures.*

Preparatory to a more extended comparison of triangles and other rectilinear figures; and a discussion of some other properties of individual polygons, it is necessary to consider the doctrine of *proportion.*

### *Of Ratios and Proportions.*

73. Before entering upon an examination of the laws of proportion, it will be proper to explain the meaning of certain *signs* which are found very convenient in expressing mathematical truths. When we wish to indicate the *addition* of two quantities expressed by A and B, we place between them this sign $+$, which is equivalent to the Latin word *plus*, signifying *more;* thus, $A + B$; which is usually read, A plus B, and means A added to B. It expresses, of course, the *sum* of the magnitudes A and B.

When we would indicate that B is subtracted from A, we use the sign —, which is equivalent to the Latin *minus*, i. e. *less;* thus, $A - B$, usually read, A minus B, signifies the remainder after subtracting the magnitude B from the magnitude A, and expresses the *difference* of the two magnitudes.

To express the multiplication of two magnitudes, we use this sign $\times$; thus, $A \times B$; which is read, A *multiplied by* B; and signifies the *product* of these magnitudes.

A period (.) placed between two magnitudes, also denotes their multiplication; thus, $A \cdot B$ signifies A *multiplied by* B.

To denote the division of one magnitude by another, we write them in the form of a fraction; thus, $\frac{A}{B}$; which is read A *divided by* B, and expresses the *quotient* arising from the division of A by B.

Instead of the expression $A \times A$, we write, for the sake of brevity, $A^2$; which signifies the product of A multiplied by itself, and is read, A *second power*. If we would indicate the second power of the sum of two magnitudes, we should write $(A + B)^2$; the second power of

the product, $(A \times B)^2$ *. To indicate the *third power* of a magnitude, we write $A^3$; and powers generally are designated in a similar manner. The figures $^2$, $^3$, &c. are called *exponents.*

To indicate any *root* of a magnitude, we write the letter indicating the magnitude with a *fractional exponent* expressing the root, as $A^{\frac{1}{2}}$, which we read, the *second root* of A. So also the *third root* of A — B, is written $(A - B)^{\frac{1}{3}}$.

To show that two magnitudes are *equal,* we write them with the sign $=$ between them; as $A = B$; this is read, A *equals* B. The expression $5 + 4 = 9$, is read 5 *plus* 4 *equals* 9. When we wish to state that A is *greater than* B, we write them thus $A > B$. If we would say that A is *less than* B, we write them $A < B$. The expressions 2A, 3A, &c. indicate *double, triple,* &c. the magnitude represented by A.

We now proceed to the consideration of ratios.

74. To find a *common measure* of two lines, and hence their *ratio,* we employ a method similar to that in arithmetic for finding the greatest common divisor of two numbers.

Let AB and CD (*fig.* 43) be two straight lines of which a common measure is required. Apply the less CD to AB the greater, as many times as CD is contained in AB; suppose this to be three times with a remainder EB; so that we shall have AB equal to three times CD added to EB, or $AB = 3CD + EB$. Now let us apply the remainder EB to CD; as it will be contained four times with the remainder FD, we shall have $CD = 4EB + FD$. Apply now the second remainder FD to EB; it will be contained once, with the remainder GB; this gives $EB = FD + GB$. Then apply GB to FD; it will be contained twice, exactly, which gives $FD = 2GB$. By retracing the steps of the operation, we shall perceive that $EB = 3GB$; that $CD = 14\,GB$; and that $AB = 45\,GB$; from which it is evident, that—*The last remainder* GB *is a common measure of the straight lines* AB and CD. As this measure Fig. 43.

* Sometimes we find them written $\overline{A + B}^2$, $\overline{A \times B}^2$, &c.

Fig. 43. GB is contained forty-five times in the first, and fourteen times in the second, we say that the line AB is to the line CD in the *ratio* of 45 to 14. This ratio we write in a fraction, $\frac{45}{14}$. *The ratio of two lines, therefore, is the number of times the first contains the second;* as the ratio of two numbers is the quotient arising from the division of the first by the second.

75. It will be easy to apply this process to any other example. We should continue the comparison of the successive remainders, till the last remainder is contained an exact number of times in the next preceding. We shall soon arrive at this result; or at least so near that the error will be too small to be recognised by the senses; in this case, we say, we have the *approximate ratio* of the two lines. This explanation, notwithstanding the imperfection of the mechanical process of comparing lines, is sufficient to give us a clear idea of what is meant by *the ratio of one line to another.*

76. *Remark.* It is not at all necessary, in expressing a ratio, that the greater magnitude should stand as the numerator of the fraction. In the example above, we said, that the ratio of AB to CD, is $\frac{45}{14}$; but the ratio of CD to AB is $\frac{14}{45}$. We say that $\frac{6}{2}=3$; that is, 6 contains 2 three times, or the ratio of 6 to 2, is equal to 3; we may say also, $\frac{2}{6}=\frac{1}{3}$; that is, the ratio of 2 to 6 is equal to *one third.*

77. When four magnitudes are such that the ratio of the first to the second, is equal to the ratio of the third to the fourth, *these four magnitudes are* said to be *in proportion.* Proportion is equality of ratios. "A PROPORTION," mathematically speaking, is a *formula, expressing the equality of two ratios.* Let A represent a magnitude which contains another magnitude $a$, as many times as the magnitude B contains $b$; these four *terms* will form a geometrical proportion. This proportion we write thus: $\frac{A}{a}=\frac{B}{b}$*. We read this proportion,—The

* A proportion has generally been written in this form; $A : a :: B : b$; and read, A is to $a$, as B is to $b$. The terms A, and B, are called *antecedents;* and $a$, $b$, are called *consequents.* The terms A and $b$ are called the *extreme* terms; $a$ and B are called the *mean* terms. A

ratio of A large to $a$ small, equals the ratio of B large to $b$ small; or, A divided by $a$ equals B divided by $b$. That is, A contains $a$ as many times as B contains $b$.

78. Suppose that A contains $a$ three times; then B must contain $b$ three times. Then the ratio of A to $a$ is equal to 3; that is $\frac{A}{a}=3$. So also $\frac{B}{b}=3$. We therefore see the propriety of the expression $\frac{A}{a}=\frac{B}{b}$. Upon this supposition we have $\frac{a}{A}=\frac{1}{3}$, and also $\frac{b}{B}=\frac{1}{3}$; therefore, on account of the common ratio $\frac{1}{3}$, we have $\frac{a}{A}=\frac{b}{B}$. If we compare this equation with the one above, we shall see that the terms of each ratio are the same, but the numerators have taken the places of the denominators, and *vice versâ*. We therefore have—I. *In any proportion, the terms of each ratio may be inverted, and the expression will still be a proportion.* This change we call *inversion.*

79. If, in the original proportion, $\frac{A}{a}=\frac{B}{b}$, we multiply the equal ratios by $a$, we shall have $A=\frac{a\times B}{b}$; if now we divide both sides by B, we shall have $\frac{A}{B}=\frac{a}{b}$. That is —II. *In any proportion the ratio of the numerators, equals that of the denominators.* This change we call *alternation.*

80. If A contain $a$ three times, $A+a$ will contain $a$ *four* times; so also, if B contain $b$ three times, $B+b$ will contain $b$ *four* times; we shall therefore have the

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fundamental principle of such a proportion is—*The product of the two means is always equal to the product of the two extremes.* The proportion, therefore, will give $A\times b=a\times B$. If we divide both sides of this equation, by $a$, and also by $b$, we shall have $\frac{A}{a}=\frac{B}{b}$, the expression in the text. It is believed that the principles of a proportion are more readily apprehended from this fractional method of writing them.

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proportion $\frac{A+a}{a}=\frac{B+b}{b}$; that is,—III. *In a proportion, the sum of the terms of the first ratio, contains its denominator as many times as the sum of the terms of the second ratio contains its denominator.*

81. How many times soever A may contain $a$, $A-a$ will contain $a$ one time less; and how many times soever B may contain $b$, $B-b$ will contain $b$ one time less. This gives $\frac{A-a}{a}=\frac{B-b}{b}$; that is—IV. *The difference of the terms of the first ratio, contains the denominator as many times as the difference of the terms of the second ratio contains its denominator.*

82. From the two last articles, we have *by alternation* $\frac{A+a}{B+b}=\frac{a}{b}$; and $\frac{A-a}{B-b}=\frac{a}{b}$; but as $\frac{a}{b}$ is common to the two proportions, and also equal to $\frac{A}{B}$ we have $\frac{A+a}{B+b}=\frac{A-a}{B-b}=\frac{A}{B}=\frac{a}{b}$. That is—V. *The sum of the terms of the first ratio of a proportion contains the sum of the terms of the second ratio as many times as the first numerator contains the second, or as the first denominator contains the second denominator.* Also—VI. *The difference of the terms of the first ratio, contains the difference of the terms of the second ratio, as many times as the first numerator contains the second, or as the first denominator contains the second*, and, consequently—VII. *The sum of the terms of the first ratio contains the sum of the terms of the second as many times as the difference of the terms of the first contains the difference of the terms of the second.*

83. If we apply the propositions of the preceding article to the formula $\frac{A}{B}=\frac{a}{b}$ (79), we shall have $\frac{A+B}{a+b}=\frac{A-B}{a-b}=\frac{A}{a}=\frac{B}{b}$. That is—VIII. *In any proportion, the sum of the numerators contains the sum of the denominators as many times as one numerator contains its denominator.* Also—IX. *The difference of the numerators contains the difference of the denominators as many times as one numerator contains its denominator.* And, there-

fore—X. *The sum of the numerators contains the sum of the denominators as many times as the difference of the numerators, contains the difference of the denominators.*

84. If we have a series of equal ratios, $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}=\frac{D}{d}=\frac{E}{e}$, &c.; by applying law VIII, we obtain $\frac{A+B}{a+b}=\frac{B}{b}$; or as $\frac{B}{b}=\frac{C}{c}$, we may write instead of this $\frac{A+B}{a+b}=\frac{C}{c}$; and applying the same law to this, we have $\frac{A+B+C}{a+b+c}=\frac{D}{d}$, and consequently $\frac{A+B+C+D+E}{a+b+c+d+e}=\frac{E}{e}$; wherefore—XI. *In any series of equal ratios, the sum of the numerators contains the sum of the denominators as many times as one numerator contains its denominator.*

85. If we have the two proportions $\frac{A}{a}=\frac{B}{b}$ and $\frac{C}{c}=\frac{D}{d}$, we can multiply them together; that is, the first ratio of the first proportion by the first ratio of the second, and the second ratio of the first, by the second ratio of the second; and as they are equal quantities, multiplied by equal quantities, they will give equal products. But to multiply two fractions together, we multiply their numerators for a new numerator, and their denominators for a new denominator. This will give $\frac{A\times C}{a\times c}=\frac{B\times D}{b\times d}$. This process is called *multiplying the proportions in order;* the result is a *compound proportion,* and each of the ratios is a *compound ratio.* We say then—XII. *Proportions multiplied in order, will give a proportion.*

86. It is evident from the last article, that a proportion may be multiplied by itself, term by term, and give a proportion; thus, the proportion $\frac{A}{a}=\frac{B}{b}$, multiplied by itself, term by term, gives $\frac{A\times A}{a\times a}=\frac{B\times B}{b\times b}$; but $A\times A$ is written $A^2$ (73), $B\times B$ is $B^2$, &c. This proportion may therefore take the form $\frac{A^2}{a^2}=\frac{B^2}{b^2}$. If we multiply

the proportion $\frac{A^2}{a^2}=\frac{B^2}{b^2}$, by the original proportion $\frac{A}{a}=\frac{B}{b}$, it will give $\frac{A^3}{a^3}=\frac{B^3}{b^3}$. As we may take the powers, so also we may take the roots (of the same degree) of the several terms of a proportion and form another proportion; thus, $\frac{A^{\frac{1}{2}}}{a^{\frac{1}{2}}}=\frac{B^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ and $\frac{A^{\frac{1}{3}}}{a^{\frac{1}{3}}}=\frac{B^{\frac{1}{3}}}{b^{\frac{1}{3}}}$. We therefore have this general law—XIII. *The same powers or the same roots, of the terms of a proportion, are also in proportion.*

87. ***Remark.*** The ratio $\frac{A^2}{a^2}$ is called the *duplicate ratio* of A to *a*; the ratio $\frac{A^3}{a^3}$ is called the *triplicate ratio* of A to *a*. The ratio $\frac{A^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ is called a *sub-duplicate ratio;* and $\frac{A^{\frac{1}{3}}}{a^{\frac{1}{3}}}$, a *sub-triplicate ratio.*

We now return to the examination of geometrical figures.

## *Of Similar Figures.*

88. In constructing the triangles &c. in the preceding operations, we have considered ourselves as taking the *absolute magnitudes* of the sides given in the problems; but in most cases in practical geometry this is inconvenient and undesirable. The methods used in surveying fields and townships, and constructing maps, afford a good illustration of this part of the subject. The lines by which the field &c. is bounded, are measured, and also the several angles; and from the *notes of the survey* a perfect representation of the outline of the field, township, &c. is constructed upon paper. This is called *a plan of the field, township, &c.* Suppose, for instance, that we have a triangular field whose sides are *eight, ten,* and *twelve chains,* respectively; the method of construction would be similar to that of the triangle with three given sides (38). But instead of taking the absolute

lengths of these known sides, we take lines embracing *eight, ten,* and *twelve* divisions upon a scale of equal parts, of any magnitude, $\frac{1}{10}$ of an inch, for instance, which will give us a plan of a convenient size, as in the figure ABC (*fig.* 44). This is a complete representation of the field surveyed; so also it would be, if the sides of the field had been eight, ten, and twelve *rods*, or *yards*, instead of chains. We should also have respresented the field as perfectly, if instead of the scale of tenths of an inch, we had used a scale divided into twentieths of an inch, as in the figure *a b c*. If then all these different triangles are faithfully represented by each other, they must have a common character; they must be *similar*. Let us see in what this similarity consists. Fig. 44.

89. 1st. The angles of the one are respectively equal to the angles of the other. It is evident that this is necessary; for if these artificial triangles have angles different from those of the field, they cannot represent its form, which is essential to a map or plan. If we had measured one side and the two adjacent angles, the side represented by AB, and the angles A and B; in drawing these two plans of the field, we should have made the angles A and *a*, each equal to that angle of the field which they represent; and the angles B and *b*, each equal to the other measured angle. And the two sides *a c*, *b c*, having the same inclinations to *a b*, as AC, BC have to AB, must have the same inclination to each other, as AC and BC have; that is, the angle *c* must be equal to the angle C; and consequently the three angles of the one must be respectively equal to the three angles of the other. This then is an essential property of similar triangles, *their angles are respectively equal.*

90. 2dly. Recurring to the construction of the two triangles, ABC, *a b c*; the divisions of the scale with which the larger was constructed, are just double the divisions of the scale with which the other was constructed; so that each side of the triangle ABC, is just double the corresponding side of the triangle *a b c*. If the divisions in the two scales had borne any other ratio to each other, the corresponding sides of the two triangles would have borne the same ratio. But the ratio of AB to *a b* being equal to the ratio of the two scales; and also the ratio of AC to *a c* being equal to the ratio of the two scales; it follows that AB contains *a b* as many

times as AC contains $ac$, that is $\frac{AB}{ab} = \frac{AC}{ac}$. In the same manner we may show that AB contains $ab$ as many times as BC contains $bc$; that is, $\frac{AB}{ab} = \frac{BC}{bc}$. These two proportions may be written together, thus, $\frac{AB}{ab} = \frac{BC}{bc} = \frac{AC}{ac}$. We see therefore, that—*In similar triangles, the corresponding or* homologous *sides are proportional.*

91. If, in making a survey of the field, we had measured one of the angles and the two sides containing that angle, the construction of the plans would have been analogous to the solution of the problem in Art. 40; and plans upon two different scales will accurately represent the field, if we construct, in each, an angle equal to the measured angle of the field, and take for the lines which represent the two sides in the larger plan, as many divisions of the larger scale as there are chains or rods in these lines respectively, and for the corresponding sides in the smaller plan the same number of divisions respectively, of the smaller scale. The corresponding sides will have the ratio of the two scales, and will therefore be in proportion. We see then, that—*Two triangles will be similar, when an angle of the one is equal to an angle of the other, and the sides containing the equal angles are proportional.*

If we draw from the vertices C, *c*, of these triangles, perpendiculars to the opposite sides, as they measure the distance of that corner of the field from the opposite side, they must have the same ratio to each other as the corresponding sides of the two plans; and so of any other similar lines in the two plans respectively; whence we derive the proposition—*In similar triangles, the homologous angles are equal, and all homologous lines are proportional.*

*Remark.* Either of these conditions, however, is sufficient to determine their similarity; for by altering the angles, we alter the ratio of the sides and *vice versâ.*

92. If the field were a rectangle, a parallelogram, or any other rectilinear figure, it might also be represented by plans upon different scales; and these plans will be similar to each other, and similar to the field, when, in

each, the angles are respectively equal to the angles of the field, and their dimensions also proportional to the corresponding dimensions of the field. Hence we have the general truth—*Two similar rectilinear figures have the angles of the one equal to the angles of the other, each to each, and all homologous lines proportional.*

93. *Remark.* If diagonals were drawn through corresponding vertices in the two polygons; being homologous dimensions, they would be proportional; and being proportional, the triangles into which these diagonals divided the two figures would be similar respectively. We say therefore, that—*Similar polygons are composed of the same number of similar triangles, similarly placed.*

94. If the several sides of a polygon were represented by A, B, C, D, E, and the corresponding sides of a similar polygon, by $a$, $b$, $c$, $d$, $e$; we should have (92) $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}=\frac{D}{d}=\frac{E}{e}$; from which we obtain (84),

$$\frac{A+B+C+D+E}{a+b+c+d+e}=\frac{A}{a};$$

that is—*In similar polygons, the perimeters are proportional to their homologous sides.*

95. In the base of the triangle ABC (*fig.* 45), take A $c$ of any convenient magnitude, and draw $c\,b$ parallel to C B; we shall have the triangle A $c\,b$, equiangular with ACB (24), and therefore similar to it (91). Hence—*If a triangle is cut by a straight line parallel to one of the sides, the small triangle cut off is similar to the whole.* Fig. 45.

These similar triangles give the proportion $\frac{AC}{Ac}=\frac{AB}{Ab}$; therefore (81), $\frac{AC-Ac}{Ac}=\frac{AB-Ab}{Ab}$; that is, $\frac{Cc}{Ac}=\frac{Bb}{Ab}$; and the line $c\,b$ divides the two sides AB and AC, proportionally. Thus we have the general rule,—*A straight line drawn through any triangle parallel to one of the sides, divides the other two sides proportionally.*

The *converse* of these two propositions may be proved from Art. 91.

96. If another parallel, $b'\,c'$, were drawn between $b\,c$ and BC, it would give the proportion $\frac{Ac'}{Ab'}=\frac{Cc'}{Bb'}$ and

$\frac{A\,c'}{A\,b'} = \frac{c\,c'}{b\,b'}$; therefore, $\frac{c\,c'}{b\,b'} = \frac{C\,c'}{B\,b'}$; that is—*When two straight lines are cut by any number of parallel lines they are divided into proportional parts; and if one of the lines is divided into equal parts, the parts of the other will be equal.*

97. To find a *fourth proportional* to three given lines
Fig. 46. (*fig.* 46). Draw the line AB equal to the sum of the first and second lines in the proportion, take AD equal to the first; and draw AC equal to the third, and making any angle with AB; join DC, and through B draw BE parallel to DC; produce AC to E; because DC is parallel to BE, we have $\frac{AD}{DB} = \frac{AC}{CE}$ (96); therefore CE is the *fourth proportional* required.

98. To find a *third proportional* to two given lines. Take a third line equal to the second; and find the fourth as above; this will be the *third proportional* required. In this case the second of the given lines is called *a mean proportional* between the first and third.

Fig. 47. 99. Let ABC (*fig.* 47) be a right-angled triangle; from the vertex A of the right angle draw the perpendicular AD to the hypothenuse BC. If we compare the triangle ABD with the original triangle, we shall see that they have each a right-angle, and the common angle B; the other angles must therefore be equal (35), viz. the angle C to the angle BAD; the triangle ABD is therefore *similar* to the triangle ABC; in the same manner it may be shown that the other triangle ADC is similar to the triangle ABC (91); whence we say—*If from the vertex of the right-angle of a right-angled triangle, a straight line be drawn perpendicular to the hypothenuse, it will divide the triangle into two triangles, each similar to the original triangle.*

100. The similar triangles ABD, ACD, give the proportion $\frac{BD}{AD} = \frac{AD}{DC}$; that is—*The perpendicular is a mean proportional between the two segments of the hypothenuse.* The similar triangle ABC, ABD, give the proportion $\frac{BC}{BA} = \frac{BA}{BD}$; that is, BA is a mean proportional between BC and BD. In the same manner we could show that CA is a mean proportional between BC and

DC; therefore—*Each side of the original triangle is a mean proportional between the hypothenuse and the segment of the hypothenuse adjacent to that side.*

We shall hereafter show the method of finding a mean proportional between any two given lines.

101. The equation $\frac{BD}{BA}=\frac{BA}{BC}$, gives $BD=\frac{(BA)^2}{BC}$; and the equation $\frac{DC}{CA}=\frac{CA}{BC}$, gives $DC=\frac{(CA)^2}{BC}$; adding these two equations we obtain $BD+DC=BC=\frac{(BA)^2+(CA)^2}{BC}$; multiplying both sides by BC, we have $(BC)^2=(BA)^2+(AC)^2$. From this it is evident that if we refer to a common measure the three sides of a right-angled triangle, the second power of the number expressing the length of the hypothenuse, will be equal to the sum of the second powers of the numbers which express the lengths of the other two sides. We can therefore find the hypothenuse when we have the other sides. Suppose in this triangle that AB contains the common measure six times, and AC eight times; then $(BC)^2=36+64=100$; and taking the second root of both sides of this equation, we have $BC=(100)^{\frac{1}{2}}=10$. So also if we know the hypothenuse and one of the sides, we can find the other. Suppose we have the hypothenuse equal to 10, and the side AB equal to 6, to find the side AC. The equation $(BC)^2=(BA)^2+(AC)^2$, by subtracting $(BA)^2$ from both sides, will give $(AC)^2=(BC)^2-(BA)^2=100-36=64$; and taking the second root of each side, we obtain $AC=(64)^{\frac{1}{2}}=8$.

102. In the triangle ABC (*fig.* 48), bisect the angle B by the line BD; and through the point C, draw CE parallel to DB till it meets AB produced. The angle ABC, exterior to the triangle CBE, is equal to the sum of the two interior opposite angles BCE and BEC (37); but on account of the parallels DB and CE, the angle ABD (half of ABC) is equal to the angle E; the other half must be equal to the angle BCE, consequently BCE and BEC will be equal; therefore, the opposite sides BC and BE are equal (45). Now in the triangle AEC, DB is drawn parallel to the base, which Fig. 48.

gives the proportion, $\frac{AB}{BE}=\frac{AD}{DC}$ (98); but BC is equal to BE, and may be substituted for it in the proportion; which gives, $\frac{AB}{BC}=\frac{AD}{DC}$; that is—*A straight line bisecting an angle of a triangle, divides the opposite side into parts proportional to the adjacent sides.*

Fig. 49. 103. Let us now take an obtuse-angled triangle, as ABC (*fig.* 49) and draw perpendiculars from the vertices of each of the angles to the opposite sides. The lines AF and BG, respectively perpendicular to BC and CA, will meet in the point D; we wish to ascertain the point M, in which CE perpendicular to BA, meets BG.

First, it is manifest that, as the two triangles ABG and EBM, have each a right angle, and the angle EBM common, they are equiangular and similar; and give the proportion $\frac{BA}{BG}=\frac{BM}{BE}$.

If we now compare the triangles EBC, and ABF, having each a right angle and the common angle ABC, they give the proportion $\frac{BC}{BE}=\frac{BA}{BF}$. And a comparison of the two triangles BGC, BFD, give the proportion $\frac{BD}{BF}=\frac{BC}{BG}$. Multiplying in order, the two last proportions, gives $\frac{BC\times BD}{BF\times BE}=\frac{BC\times BA}{BF\times BG}$; if now we divide each ratio of this proportion by the common multiplier $\frac{BC}{BF}$, we have $\frac{BA}{BG}=\frac{BD}{BE}$; and comparing this proportion with the first, $\frac{BA}{BG}=\frac{BM}{BE}$; we perceive that they have the common ratio $\frac{BA}{BG}$; the other two ratios are therefore equal, and give the proportion $\frac{BM}{BE}=\frac{BD}{BE}$; that is, BM is equal to BD, and the three perpendiculars meet at the same point D.

If the original triangle were acute-angled, as ABD in the same figure, it is manifest that the three perpen-

diculars drawn from the vertices to the opposide sides, would meet in the point C. We say therefore—*In any triangle, the perpendiculars drawn from the vertices of the angles to the opposite sides, will meet in the same point.*

*Remark.* If the triangle be acute-angled, the point of meeting will be *within* the triangle; if the triangle be obtuse-angled, the point of meeting will be *without*; and in the right-angled triangle, the point of coincidence is the vertex of the right-angle.

### *Of the Straight Line and Circle.*

104. The *circle* is the plane surface embraced by the circular curve already described (7), called the circumference of the circle. A straight line which touches the circumference only in one point, is called *a tangent*, as AB (*fig.* 50). When it is said that Fig. 50.
the straight line *touches* the circumference, it is meant that its direction is such that if it were produced ever so far each way, it would have only one point in common with the circumference. This point is called the *point of contact.*

105. If the straight line cuts the circumference in two points, it is called *a secant.* A straight line which has its two extremities in the circumference, as DE (*fig.* 51), Fig. 51.
is called *a chord;* it *subtends* both the arc and the angle at the centre measured by this arc. The portion of *surface* contained between the two radii DO, EO, and the *arc* DGE, is called *a sector.* The portion contained between the chord and the arc is called *a segment of the circle.*

106. If the two radii DO, EO, the chord DE and the arc DGE, be made to revolve together about the centre O, of the circle, the points D and E will be continually in the circumference; and the arc DEG will always coincide with the part of the circumference then embraced by the two radii (7); if this were not the case, some parts of the circumference must be nearer to the centre than other parts; moreover, the straight line DE would be constantly the chord of this arc. Now as the angle DOE would not be changed by this revolution of the sector DOE, we say that this angle would have the same arc in every part of the circumference, and this arc would also have the same chord. We have, therefore,

these propositions:—(1). *In the same circle or in equal circles, equal angles at the centre are measured by equal arcs; and equal arcs subtend equal angles.* (2). *In the same circle, or in equal circles, equal arcs are subtended by equal chords; and when, in the same circle, the chords are equal, the arcs are equal.*

As the triangle would not be changed by this revolution, the distance of the vertex O from the opposite side DE, would not vary; that is—*In the same circle, equal chords are at the same distance from the centre.*

107. If the arc were increased, the angle at the centre would be increased (26); but the two sides of the triangle OD and OE remaining the same, if their angle is increased, the third side DE must be increased (32); therefore—*The greater arc is subtended by the greater chord.* By increasing the side DE of the triangle, we increase the angle O, and the arc which measures this angle; therefore—*The greater chord subtends the greater arc.*

108. We said that the chord increases as the arc is increased; this is true to a certain extent; that is, while the arc is less than 180°, or a semi-circumference. At this point the *chord* is confounded with the *two radii*, and becomes a diameter (7); it subtends no angle, and is no longer called a chord. But as the chord continues to increase till it is confounded with the diameter, we see that—*Of two unequal chords, the greater is at the less distance from the centre.*

If the arc be increased beyond 180°, the two radii will make an angle with each other on the other side, and the chord which subtends this arc, will make the third side of the triangle, and is therefore less than the sum of the other two (50), and consequently less than the diameter. Therefore—*In the same circle a chord is always less than the diameter.*

It will be perceived that *every chord corresponds to two arcs*, which taken together constitute the circumference.

Fig. 52. 109. In *fig.* 52, suppose the radius CD to be drawn perpendicular to the chord AB; if we draw the other two radii CA and CB, we shall have the isosceles triangle ACB; CE perpendicular to the base AB, bisects the base and the angle at C (45); as the two angles at C are equal, the arcs AD and DB are also equal (107); whence we say—*Radius perpendicular to a chord, bisects the chord and its arc.*

As one point and the direction of a straight line, determine its position (53)—*A straight line drawn perpendicularly through the middle of a chord, bisects the arc, and passes through the centre of the circle.*

110. This enables us *to find the centre of a circle to which any arc belongs (fig.* 53). We have only to assume two chords, AD, DB, in the arc, and draw perpendiculars through the middle of each; the point of meeting of these two lines is the centre of the circle. Fig. 53.

The operation would be the same if we were required to draw a circular curve through the *three* given points A, D, B. If, however, these three points were in the same straight line, it is manifest that the perpendiculars would be parallel (23); they would therefore nevèr meet; consequently—*No three points in the same straight line, can be in the circumference of a circle.*

Can you make two or more circumferences pass through three given points not in the same straight line?

Can you make a circumference pass through *any four* points not in the same straight line?

111. As the tangent has but one point in common with the circumference, namely, the point of contact (*fig.* 54), every other point must be without, and consequently farther from the centre of the circle; and if radius be drawn to this point, it will be less than any other straight line drawn from the centre to the tangent; it must consequently be perpendicular to the tangent (51); therefore—*To draw a tangent to any point in a circumference, we have only to draw a straight line perpendicular to radius at that point.* Fig. 54.

As many circles, therefore, as have the point A, a point in the circumference, and BD, a common tangent, must have their centres in the same straight line CA, produced if necessary. In this case *the circumferences are* said to be *tangent to each other.*

When the circles have their centres on *opposite* sides of the point of contact, they are said to be *tangent externally.* When they have their centres on the same side, they are said to be *tangent internally.*

Can a straight line pass through the point of contact between a circumference and a rectilinear tangent, which shall cut neither of them?

Can you draw a circular line between a right-line tangent and the circumference?

Can you draw a circular line between two circular curves tangent to each other, and having their centres on the same side of the point of contact?

Fig. 55. 112. If we draw the secant, MN (*fig.* 55) parallel to the tangent, it will be perpendicular to the radius CA; and as the interior portion of the secant, will be a chord, the arc which it subtends will be bisected by this radius (109) at the point of contact. So if another parallel secant OP be drawn, the arc which this cuts off will also be bisected by the same radius; so that we shall have the arc AO equal to the arc AP; and AM equal to AN; therefore MO and NP are equal. Hence—*Parallels intercept equal arcs in the circumference.*

113. We have seen (106) that equal angles at the centre of a circle, are subtended by equal arcs; if, therefore, we find a common measure of the two arcs AB, A′B′, (*fig.* 56) drawn with equal radii; and divide each of the

Fig. 56. arcs by this common measure; and to the several points of division draw radii; the two angles ACB, A′C′B′, will be divided into equal angles. If the arc $a\,b$ is a common measure of the arcs, and the angle $a\,c\,b$, a common measure of the two angles; as there will be in each of the sectors as many of the small arcs as angles; we shall have $\frac{\text{ACB}}{a\,c\,b}=\frac{\text{AB}}{a\,b}$ and $\frac{\text{A}'\text{C}'\text{B}'}{a\,c\,b}=\frac{\text{A}'\text{B}'}{a\,b}$; as the denominators are the same in both proportions the numerators must be in proportion; which gives (79) $\frac{\text{A}'\text{C}'\text{B}'}{\text{ACB}}=\frac{\text{A}'\text{B}'}{\text{AB}}$; that is, in circles drawn with equal radii, the angles at the centre are to each other in the ratio of the arcs which subtend them, if these arcs are commensurable.

Suppose the arcs have no common measure; we may take a measure of one of them, A′B′ for instance (*fig.* 57), so small that by applying it to AB as many times as

Fig. 57. it is contained in AB, the remainder $b$ B which we will designate by $m$, will be less than any assignable arc; and the angle $b$ CB, which it subtends, less than any assignable angle, which indefinitely small angle we designate by $n$. The arcs A′B′ and A $b$ are commensurable, and therefore give the proportion $\frac{\text{ACB}-n}{\text{A}'\text{C}'\text{B}'}=\frac{\text{AB}-m}{\text{A}'\text{B}'}$.

Now suppose that, to AB produced we apply this small measure of A'B', once more than before, it will extend to $b'$; and suppose also that the excess of this arc A $b'$ over *AB*, is less than any assignable arc, which excess we designate by $m'$, and the corresponding indefinitely small angle by $n'$; we shall have $\frac{ACB+n'}{A'C'B'}=\frac{AB+m'}{A'B'}$. It is manifest, therefore, from these proportions, that, one of the arcs, with its angle, remaining the same, if any magnitude however small be subtracted from the other arc, a corresponding magnitude must be subtracted from its angle, to preserve the proportion; and if any magnitude however small be added to this arc, a corresponding magnitude must be added to its angle to preserve the proportion; from which it follows that if nothing be either added to or subtracted from the one, nothing must be either added to or subtracted from the other. We shall therefore have the proportion, $\frac{ACB}{A'C'B'}=\frac{AB}{A'B'}$, even when the arcs are incommensurable. We therefore say—*If from the vertices of two angles, arcs are described with the same radius, the portions of these arcs embraced by the sides, will be in the ratio of the angles.*

114. We see from the preceding article, what is to be understood when it is said that—*An angle has for its measure the circular arc comprehended between its sides, the vertex being at the centre of the circle.* By this we mean that the angle is the same part of *four right-angles*, as the arc is of an entire circumference or 360°. When therefore, we speak of an angle of 30°, we mean an angle which would embrace between its sides, 30 parts out of 360, into which any circumference described from the vertex of this angle as a centre, should be divided.*

115. Suppose that the vertex of the angle, instead of being at the centre of the circle, is at some point between the centre and the circumferance, as at C (*fig.* 58). If Fig. 58.

* This is according to the ancient division of the circumference, mentioned in article 26; a different system has been adopted in France. This system divides the circumference into 400 degrees; each degree into 100 minutes, and each minute into 100 seconds.

we draw through the centre, the two lines $a\ n$ and $b\ m$, parallel to the two lines AN and BM, they will make an angle at the centre equal to the angle to be measured (24); and the arc $m\ n$ will be as much greater than MN as the arc $a\ b$ is less than AB; so that the *sum* of the arcs intercepted by the lines passing through the centre, is equal to the sum of the arcs intercepted by the lines which meet at C; therefore the arc $a\ b$ is equal to *half the sum of* the two arcs AB and MN. But $a\ b$ measures the angle $a$ O $b$ equal to ACB. Whence we have this truth—*An angle whose vertex is in the interior of a circle, made by two chords, has for its measure half the sum of the intercepted arcs.*

116. Suppose the vertex of the angle to be in the circumference (in which case the angle is called an *inscribed angle*). By drawing parallels to the chords through
Fig. 59. the centre (*fig.* 59) we shall have the two arcs C $n$ and C $m$, equal to A $a$ and B $b$; therefore the arc $m\ n$ equals A $a$ + B $b$; consequently the arc $a\ b$ equals A $a$ + B $b$; that is, the arc $a\ b$ is half the arc AB; but $a\ b$ measures the angle whose vertex is at O, equal to the inscribed angle.

When both the chords which form the angle are on the same side of the centre, as the angle ACB′; this angle may be considered as the difference of the two angles BCB′, BCA; BCB′ has for its measure half the arc BB′; BCA has for its measure half of AB; consequently, the remaining angle ACB′ must have for its measure half the remaining arc AB′; and therefore—*Any inscribed angle has for its measure half the arc embraced by its sides.*

117. This result includes the case of the angle made by the chord and tangent; for if AC be made to revolve about the point C till it come into the position A′C; as the angle ACB increases, the arc AB increases, half of it still measuring the angle; and this reasoning holds till AC come infinitely near to the position of the tangent. We infer, therefore, that the error, if there be one, must be infinitely small.

But to ascertain whether there is any error in this result, draw through the centre parallels to the chord and
Fig. 60. tangent (*fig.* 60); the angle ACB will be equal to $a\ c\ b$, measured by the arc $a\ b$. On account of the parallels, the arcs C $a$ and C $n$ are equal; and we have the arc $m\ n$

equal to C *a* added to B *b;* but the arc *m n* is equal to *a b;* therefore the arc *a b* is equal to *a* C added to B *b*, that is, equal to half the arc C *a* B. Hence we have rigorously demonstrated that—*The angle made by a tangent and chord, has for its measure half the included arc.*

118. If the meeting of the two lines is beyond the circumference (*fig.* 61), draw M *b* parallel to CB; the arc A *b* will be equal to the difference between the two arcs AB and MN; and the angle AM *b*, will be equal to the angle C (24); but the angle AM *b* has for its measure half the arc A *b* (116); therefore—*The angle made by the two secants meeting without the circumference, has for its measure half the difference of the two arcs embraced by the sides.* Fig. 61

119. This includes the case of the angle made by two tangents; for drawing the chord A *b* (*fig.* 62) from one point of contact parallel to the other tangent; the arcs AB and B *b*, will be equal (112), and the arc A *b* will be the difference between the two arcs A *b* B and AB. But half this arc A *b* measures the angle DA*b* (116), and conquently the angle ACB, equal to DA*b*. Hence we say—*The angle made by two tangents, is measured by half the difference of the arcs between their points of contact.* Fig. 62.

120. From the reasoning in article 116, it follows that —*All inscribed angles subtended by the same arc, are equal* (*fig.* 63), as each has for its measnre half this arc. Fig. 63

121. If the arc embraced by the angle is a semi-circumference, that is, if the two sides of the *inscribed angle* are in the extremities of a diameter (*fig.* 64), the *semi-arc*, which measures the angle, will be an arc of 90°; therefore (27)—*An angle inscribed in a semicircle is a right-angle.* Fig. 64.

122. Two angles, inscribed in opposite segments of the same circle, as ACB, ADB (*fig.* 63), will have, each for its measure, half the opposite arc; therefore the two will have for their measure, half the circumference; consequently—*The sum of two angles inscribed in opposite segments of the same circle, is equal to two right-angles.* Fig. 63.

123. As every tangent to the circumference of a circle, is perpendicular to the radius whose extremity is the point of contact (111); if we were required to draw a tangent to the circumference BFD (*fig.* 65) from the point A without the circle; we should draw a straight line from Fig. 65.

4*

the point A to the centre C, and make AC the diameter of a circle whose circumference cuts that of the given circle in two points B, D, to which we should draw the straight lines AB, AD; these will be tangents to the circumference BFD. For, drawing the radii CB, CD, the angles ABC, ADC, being each inscribed in a semicircle, will be right angles; that is, the straight lines AB, AD, are perpendicular to the radii CB, CD, at their extremities (121). It appears that, from any point without a circle, *two* tangents may be drawn to that circle.

Fig. 66. 124. If we have the straight line FG given in position (*fig.* 66), and the point B also given, we can describe a circumference passing through B, to which GF shall be tangent at the point A. Draw AB; bisect this line by the perpendicular DC; and from A draw a line perpendicular to FG, meeting DC in the point C; this point will be the centre, and AC the radius of the required circumference.

If BA be produced to E, the *acute* angles at A, made by the tangent and chord produced, will be equal to the angles inscribed in the *greater* segment of this circle; the *obtuse* angles at A, made by the tangent and chord produced, will be equal to the angles inscribed in the *less* segment (116, 117). It is therefore easy to solve this general problem,—*Upon a given straight line* (as AB in this figure) *to describe the segment of a circle, in which a given angle may be inscribed.*

125. We have seen how we are to measure the *angles* which the *chords*, *tangents*, and *secants* of any circle make with each other. Let us inquire into the laws which regulate the proportion of their parts.

Let AB and CD be two *chords* which cut each other
Fig. 67. at E (*fig.* 67). By drawing the chords CA and BD, we have two triangles CEA and BED; the two angles at C and B are equal, because they have each for a measure half the arc AD; the angles at E are *vertical angles*, and are therefore equal; consequently the other two angles A and D, are equal, and the two triangles are therefore similar, and give the proportion, $\frac{AE}{EC} = \frac{ED}{EB}$. That is, the greater part of each chord divided by the less part of the other, gives the same ratio. This is what is called *an inverse* or *reciprocal* proportion; and we say—

*Two chords which intersect, divide each other into parts reciprocally proportional.*

126. If these chords are perpendicular to each other, and one of them (as AB *fig.* 68) is a diameter, it will bisect the other in E (109); and therefore in the proportion, instead of ED, we may use its equal EC. This will give $\frac{AE}{EC} = \frac{EC}{EB}$; that is, EC is a mean proportional between AE and EB. Fig. 68.

Suppose we draw also the chord BC; since AB is a diameter, the angle ACB inscribed in a semicircle, is a right angle; and CE is a perpendicular drawn from the vertex of the right-angle of the right-angled triangle ABC, to the hypothenuse, dividing the triangle into two smaller triangles.

If we examine the triangles into which the triangle ACB is divided, we shall perceive that the triangle ACE has a right angle and the angle A in common with the larger triangle; the triangle BCE has a right angle, and the angle EBC in common with the larger triangle. Each of these partial triangles is therefore equiangular with the larger; they are consequently similar to it and similar to each other. If we compare each of them with the larger we have the proportions $\frac{AE}{AC} = \frac{AC}{AB}$, and $\frac{EB}{CB} = \frac{CB}{AB}$; that is, AC is a mean proportional between AB and AE; and CB is a mean proportional between AB and EB.

If, therefore, we consider these lines AC, AB, CB, and CE, in their connexion with the circle, we should enunciate in *this* manner the truths just proved; (1). *A perpendicular drawn from any point in the circumference of a circle to the diameter, is a mean proportional between the two segments of the diameter;* (2). *If, from the vertex of an angle inscribed in a semicircle a perpendicular be drawn to the diameter, each of the chords which form this angle, is a mean proportional between the whole diameter and the segment adjacent to that chord.*

The above result affords us the means of giving the solution which we promised (101) of the problem—*To find a line which shall be a mean proportional between two given lines.*

[The learner will perceive that there are here two methods of solving this problem; he can give examples of each.]

127. If, however, we regard the triangle ACB, independently of its connexion with the circle, these truths will be enunciated thus:

*If from the vertex of the right-angle in a right-angled triangle, a perpendicular be drawn to the hypothenuse,—*

(1). *This perpendicular will be a mean proportional between the two segments of the hypothenuse; and will divide the triangle into two triangles similar to each other and similar to the whole.*

(2). *Each of the sides will be a mean proportional between the entire hypothenuse and the segment adjacent to that side.*

[These propositions were differently obtained in articles 99 and 100.]

128. If these lines instead of meeting within the circumference, were to meet beyond the circumference, as
Fig. 69. in figure 69; by drawing the chords CA and BD, we have also, two equiangular triangles AEC and DEB; the angle E is common to the two triangles, and the two angles at A and D, are measured by half the same arc; consequently they are equal, and the other two angles are equal, and the triangles similar; they will give us this proportion, $\frac{EA}{EC} = \frac{ED}{EB}$; that is—*When two secants meet beyond the circumference, the entire secants will be in the inverse ratio of the parts without the circle.*

129. As the truth of this last proposition does not depend upon any particular position of the secant, we
Fig. 70. may suppose the secant DE (*fig.* 70) to revolve about the point E till the interior part CD vanishes and the secant becomes a tangent; the proposition will hold true to the vanishing point of the secant; but at that point the exterior part becomes equal to the whole; and we shall have $\frac{AE}{EF} = \frac{EF}{EB}$; that is—*If, from the same point without a circle, a tangent be drawn and also a secant, the tangent will be a mean proportional between the entire secant and the exterior part.*

By drawing the two chords AF and BF, the two similar triangles AEF and BEF, will give the same proportion. So that we are sure of the accuracy of this result.

Why are the triangles AEF and BEF similar?

130. If we make the tangent AB equal to the diameter of the circle (*fig.* 71), draw the secant through the centre, and take upon the tangent the part AF equal to the exterior part of the secant, the last article will give the proportion Fig. 71.

$$\frac{AE}{AB} = \frac{AB}{AD}; \ (81.) \ \frac{AE - AB}{AB} = \frac{AB - AD}{AD}.$$

But AE — AB is AD, and AB — AD is BF; the proportion therefore becomes $\frac{AF}{AB} = \frac{BF}{AF}$; and by inversion (78) we have $\frac{AB}{AF} = \frac{AF}{BF}$; that is, the line AB is divided at F, so that the part AF is a mean proportional between the whole line and the other part BF. This is called dividing a line into *mean and extreme ratio.*

To divide the line AB into mean and extreme ratio, draw through one extremity a perpendicular equal to half the line; make this the radius of a circle, through the centre of which draw a secant meeting the given line in the other extremity; the exterior part of the secant transferred to the given line, gives us the point of division F.

131. If we make the perpendicular BC equal to the given line AB (*fig.* 72), the angle at A will be an angle of 45°, that is, half of a right-angle (35), and the portion AC will be the diagonal of a sqnare whose side is AB. This suggests a method of comparing the diagonal with the side of the square. Fig. 72.

The method of making this comparison was given in article 74. If the side be applied to the diagonal, it will be contained in it once with a remainder AD; if then we apply AD to AB, it will be contained in it twice with some remainder. Now we know that AB contains AF as many times as AE contains AB; so that instead of comparing AF with AB, we have only to compare AB with AE; but AE contains AB twice, with the remainder AD. This remainder is to be compared with AB, which is the same thing as comparing AB with AE, but this is the same as the last. It appears, then, that this is the commencement of a series of operations which can never terminate; therefore—*The side of a square and its diagonal are incommensurable magnitudes*—a result which we shall obtain hereafter by a different method.

### *Of Polygons inscribed in a Circle, and circumscribed about a Circle.*

132. In figure 68, we have the right-angled triangle ACB, whose vertices are in the circumference of a circle of which the diameter forms the hypothenuse of the triangle; and as the vertices of *any* triangle are three points not in the same straight line (110), it is manifest that a circumference may be described passing through these vertices. Under such circumstances the triangle is said to be *inscribed in the circle.*

Fig. 73. 133. If we examine the inscribed triangle ABC (*fig.* 73), we shall perceive, that as each angle is an inscribed angle, it will have for its measure half the opposite arc, of which the side opposite to this angle is the chord; and as the greater chord subtends the greater arc (107), we have an additional proof that the greater angle in a triangle is opposite to the greater side (47). And as equal chords have equal arcs, and equal arcs subtend equal angles (106), an equilateral triangle must have its angles equal (46); and the equal sides in an isosceles triangle, must be opposite to equal angles (45).

We have also a new proof that the sum of the angles in every triangle, is equal to two right angles (35); for as each of the angles is measured by half of the opposite arc (116), the sum of three angles will have for their measure half the sum of the three arcs, or half the circumference; which is the measure of two right angles (27).

Fig. 74. 134. To inscribe a circle in a triangle; bisect two of the angles, as A and B (*fig.* 74); from the point of meeting of the two bisecting lines, draw perpendiculars OD, OE, OF, to the sides of the triangle; the point O will be the centre, and these perpendiculars will be the radii of the circle. The two triangles AOE, AOF, have each a right-angle, and their angles at A equal; their other angles will therefore be equal; and having the common side AO, they will be equal by the third case of equal triangles; OE and OF are therefore equal. In the same way it may be shown that OE and OD are equal. The circle, therefore, which has O for its centre and OE for its radius, will have each side of the triangle ABC tangent to its circumference. The circle is said

to be ***inscribed*** in the triangle; and the triangle is *circumscribed about* the circle.

135. It will be readily seen that no circumference passing through the three points A, B, C, can also pass through the point E (*fig.* 73). A quadrilateral, therefore, cannot always be inscribed in a circle. It may be well to inquire under what circumstances a quadrilateral may be inscribed in a circle. If we consider the inscribed quadrilateral ABCD, we shall perceive that the angle ADC has for its measure half the arc ABC; if the angle ADC were less than it is, the others remaining the same, its vertex would be beyond the circumference; if it were greater, the vertex would be within the circumference; it must therefore be such as to be measured by the arc ABC. So also the three points A, D, C, remaining the same, the angle at B must be such as to have for its measure half the arc ADC. But these two angles have for their measure half of an entire circumference; their sum, *therefore*, must be equal to two right angles. In the same manner we may show that the sum of the other two angles is equal to two right angles; whence we infer, that—*A quadrilateral may be inscribed in a circle, when its opposite angles are supplements of each other.* Fig. 73.

A square, therefore, and any other rectangle, may be inscribed in a circle; but no other parallelogram.

136. In polygons, generally, of a greater number of sides, it would be more difficult to ascertain whether they could be inscribed.

Suppose, however, we take a certain number of equal lines, as AB, BC, CD and DE (*fig.* 75), and dispose them as in the figure, so as to make equal angles whose vertices are at B, C, D. The circumference of only one circle can be drawn through the three points A, B, C; the relative position of these points, therefore, determines the curvature. But on account of the equality of the lines and angles, the relative position of any three contiguous vertices will be the same; the same curvature, therefore, would be required throughout, and the circular arc which passes through the three vertices A, B, C, if produced will pass through the other vertices. We say, therefore, that—*Any polygon which has its sides equal and its angles equal, may be inscribed in a circle.* Fig. 75.

***Remark.*** As the circumference of a circle may be divided into unequal parts, and chords may be drawn to

to the points of division, forming inscribed polygons; it will be perceived that there is an endless variety of polygons which *may* be inscribed in a circle.

Fig. 76. 137. If we construct the isosceles triangle ABC (*fig.* 76), the angle ACB being any *measure* of four right-angles; the perpendicular C $a$ will bisect the side AB and measure the distance of the point C from this side. If this triangle be made to revolve about the side BC, till the point A comes again into the same plane, it will occupy the place of the isosceles triangle BCA′, equal to ABC; the perpendicular C $a$ will fall upon C $a'$, which is therefore equal to C $a$; we have also the angle ABA′ double the angle ABC, the angle at the base of the triangle. If we now revolve this triangle about the line CA′ till B meets the plane of the paper in B′, we shall have C $a$ falling upon C $a''$ perpendicular to A′ B′ and therefore measuring the distance of the point C from this line; we have a polygonal angle BA′B′ equal to twice the angle at the base of the isosceles triangle, and therefore equal to ABA′. If we continue this process, as the angle ACB is a measure of four right-angles, the triangle after a certain number of turns, will have occupied all the angular space about C, bringing one of its sides to the line AC, whence the triangle first departed. We shall therefore have the polygon ABA′B′A″B″ of which all the sides are at the same distance from C; if, therefore, from the point C, with a radius equal to this distance C $a$, we describe a circumference, it will *touch* each of the sides of the polygon; the polygon is therefore *circumscribed about* the circle. If we examine this polygon we shall see that the sides are all equal, and all the angles at the vertices are equal, being each equal to twice the angle at the base of the isosceles triangle used in *generating* this polygon. We say therefore—*A polygon, whose sides are equal and whose angles are equal, may be circumscribed about a circle.* Because CA, CB, CA′, &c. are sides of the same isosceles triangle, they are all equal; and if from C as a centre with the radius CA, we describe a circumference, it will pass through the vertices of the polygon. The polygon will then be inscribed.

Which of the several classes of parallelograms (62) can be inscribed in a circle?

Which of the several kinds of parallelograms can be circumscribed about a circle?

138. In the polygons whose sides are equal and angles equal, whichever side be taken for the base, the base will be of the same magnitude, the angles at the base will be the same, and the opposite vertices will be determined by equal triangles similarly situated. On account of this regularity of their parts, these figures are called *regular polygons.* We have seen (136, 137) that polygons whose sides are equal and angles equal, may be iuscribed in a circle, and also circumscribed about a circle. We say, therefore—*Regular polygons may be inscribed in a circle, and also circumscribed about a circle.*

139. In the regular hexagon ABA′B′A″B″ (*fig.* 76), Fig. 76.
the point C is called the *centre of the polygon;* the equal angles ACB, BCA′, &c. are called *angles at the centre.* As regular polygons of the same number of sides must have equal angles at the centre (15), they will be composed of the same number of similar isosceles triangles (46), and will therefore be similar polygons (93), and as the sides of the isosceles triangles of which they are composed, are the radii of the circumscribed circles, and the heights of these triangles, the radii of the inscribed circles; it follows (92) that—*In regular polygons of the same number of sides, the perimeters are to each other as the radii of the circumscribed circles, and also as the radii of the inscribed circles.*

140. Each angle of an equilateral triangle is an angle of 60° (40); the chord of 60° is equal to radius. Therefore (*fig.* 77)—*To draw an equilateral triangle,* Fig. 77.
*describe a circular arc with a radius equal to the given side, take a chord equal to this radius, and draw radii from the centre to the extremeties of this chord.*

141. It will be perceived that, as radius is equal to the chord of 60°, it may be applied six times to the circumference. This will give us a regular, inscribed *hexagon,* as ABDEFG. If we draw chords to the alternate angles of the hexagon, we shall have an inscribed equilateral triangle. And if we bisect each of the arcs subtended by the sides of the hexagon, and draw chords to the same arcs, we shall have a regular polygon of twelve sides, called a *dodecagon.* By continuing the process, we obtain regular inscribed polygons of 24, 48, 96, &c. sides.

142. If we draw two diameters perpendicular to each other (*fig.* 78) and draw the chords (AB, BC, &c.) to Fig. 78.
their extremities, we shall have an inscribed quadrilat-

Fig. 78. eral, the sides of which are equal. We perceive that this figure is composed of four right-angled triangles, which are also isosceles; the acute angles therefore are each equal to half a right-angle; but each angle of the inscribed quadrilateral is composed of two of these equal angles, and is therefore a right-angle; consequently the figure is *an inscribed square.*

If we bisect the arcs AB, BC, &c. and draw the chords of these semi-arcs, we shall have *an inscribed octagon.* And by continuing the process of subdividing the arcs and drawing chords to them, we shall have inscribed figures of 16, 32, 64, &c. sides.

143. In the right-angled triangle AOB, we have (101) $(AB)^2 = (AO)^2 + (OB)^2$, or $(AB)^2 = 2(AO)^2$, as AO is equal to OB; and if we extract the second root of both sides of this equation, we shall have $AB = AO \times (2)^{\frac{1}{2}}$; or considering the radius AO as *unity* or one, it gives $AB = (2)^{\frac{1}{2}}$. This is the side of the inscribed square. It is always the diagonal of a square whose side is AO, equal to unity. Now it is known that there is no numerical expression for this value $(2)^{\frac{1}{2}}$ independent of the *radical form;* and as this quantity is incommensurable with unity, it follows that the diagonal is incommensurable with the side of a square (131). We see, however, that notwithstanding this incommensurableness, the geometrical process gives us their ratio exactly. *The ratio of the diagonal to the side of the square, is* $(2)^{\frac{1}{2}}$.

The side of the inscribed equilateral triangle is obtained from figure 77, thus: $(EB)^2 = (EA)^2 - (AB)^2$; but as AB is equal to radius or *one*, and EA equals twice radius or *two*, we have $(EB)^2 = 4 - 1 = 3$; therefore $EB = (3)^{\frac{1}{2}}$

The side of the regular inscribed hexagon equals radius, or one. Having the side of an inscribed polygon, we have its perimeter, and radius entering into the expression, we have the ratio of the perimeter to radius.

Fig. 79. 144. If in the circle ABFGH, &c. (*fig.* 79) we divide the radius AC into extreme and mean ratio at E, EC being the larger part, take the chord AB equal to EC, and join EB and BC; the result of this division of radius gives $\frac{AC}{EC} = \frac{EC}{AE}$; but substituting for EC its

equal AB, we have $\frac{AC}{AB}=\frac{AB}{AE}$. We have, therefore, in the two triangles ABC, ABE, the angle A common and the sides about this angle proportional; they are therefore similar (91); but as ABC is an isosceles triangle, ABE must be an isosceles triangle, and EB is equal to AB or to EC; the triangle EBC is therefore isosceles, and the angle ECB, which, on account of the similar triangles, is equal to ABE, is also equal to EBC; consequently each of the angles at the base of the isosceles triangle ABC is double the angle at the vertex; whence the angle ACB is $\frac{1}{5}$ of two right angles or $\frac{1}{10}$ of four right angles; the arc AB which subtends this angle, must therefore be $\frac{1}{10}$ of the circumference; and the chord AB is the side of a regular inscribed *decagon.*

By joining the alternate vertices BG, GI, &c. of the regular inscribed decagon, we have the regular inscribed *pentagon.* And by bisecting the arcs and drawing chords, we obtain regular inscribed polygons of 20, 40, 80, &c. sides.

145. The arc subtended by the side of a regular inscribed decagon, is one tenth of the circumference; and the arc subtended by the side of the regular inscribed hexagon, is one sixth of the circumference. Let the arc BP be one sixth of the circumference; if we subtract from this the arc BA, which is $\frac{1}{10}$ of the circumference, we shall have the arc AP $=\frac{1}{6}-\frac{1}{10}=\frac{1}{15}$ of the circumference; and the chord AP will be the side of a regular inscribed polygon of 15 sides. By the process of bisecting the arcs, we obtain polygons of 30, 60, &c. sides.

146. If we could divide any arc into three equal parts, we could obtain successively the arcs subtended by the sides of regular inscribed polygons of 9, 18, 27, 36, 45, &c. sides; but this division refers itself to the famous problem of the *trisection of the angle,* whose solution was so much sought by the elder geometers; "being in point of difficulty, or rather perhaps of impossibility, on a footing with the other two celebrated problems, viz. the duplication of the cube, and the quadrature of the circle." Elementary geometry has not yet furnished us with the means of solving this problem.

147. *Having a regular inscribed polygon of any number of sides, we may circumscribe about the circle a regular polygon of the same number of sides,* by drawing tan-

gents to the circle through the vertices of the inscribed polygon. If we draw the radii CA, CB, CD, &c. (*fig.*
Fig. 80. 80) to the vertices of the inscribed polygon ABDEFG, and draw through these vertices, perpendicular to radius, the straight lines *ab*, *bd*, *de*, *ef*, *fg*, *ga*, we shall have the regular circumscribed polygon *abdefg*, of the same number of sides.

*Remark.* The perpendicular CA, CB, &c. drawn from the centre of the regular polygon to one of its sides, (which always bisects that side) is called the *apothegm* of the polygon. C*m* is the apothegm of the inscribed polygon.

148. It is manifest that by increasing the number of sides in the inscribed polygon, we increase its perimeter, which can never be greater than the circumference; and by increasing the number of sides of the circumscribed pologon, we diminish its perimeter, which can never be less than the circumference. The circumference of the circle being then the *limit* of this approximation, or the value to which these two magnitudes approach; we say that—*The circumference of the circle is equivalent to the perimeter of the regular inscribed (or circumscribed) polygon of an infinite number of sides.*

Two circles may therefore be considered two similar polygons; whence (139)—*The circumferences of two circles are to each other in the ratio of their radii, or of their diameters.*

*Remark.* The apothegm of the inscribed or circumscribed polygon of an infinite number of sides must be infinitely near to radius.

The consideration of the ratio of the circumference to radius, or to the diameter, we shall defer for the present.

## PART FIRST.

### SECTION II.—*Of the Measure and Comparison of Plane Surfaces.*

149. In discussing the properties of plane figures thus far, we have considered only the lines and the angles, and their several relations to each other, without regarding the quantity of surface embraced by the outline or perimeter of the figure. The whole amount of surface or *superficies* in any geometrical figure is called its *area.* The word *surface,* in general, we use to signify superficial extent without regard to quantity.

150. In speaking of figures as equal, we have said that it is an indispensable condition of geometrical equality that the figures compared should coincide by superposition. But it is evident that two figures may have the same amount of surface, and still be very different in form. Such figures we call *equivalent.* A field of a circular form may have the same superficial extent with another field whose form is quadrangular; we should say that the two fields are *equivalent* or *equal in area.*

In comparing parallelograms we consider one side as the *base* of the figure, and the perpendicular distance of this side from the opposite, the *height* of the figure. Sometimes we call one side the *inferior* base, and the opposite, the *superior* base. In triangles one side is taken for the *base,* and the perpendicular distance of this side from the vertex of the opposite angle, is the *height* of the triangle.

151. To compare two parallelograms ABCD, ABEF, (*fig.* 81) whose bases are equal, and heights equal, place Fig. 81
their bases together as in figure 82. On account of their Fig. 82.
equal heights and the parallelism of their opposite sides, their superior bases CD, EF, will be in the same straight line CF. We shall have two triangles CAE, DBF, of which the sides CA and DB are equal and parallel, being opposite sides of the same parallelogram (59); AE and BF are equal and parallel for the same reason. The two angles CAE, DBF, having their sides parallel and directed the same way, are equal; and the two triangles CAE, DBF, are equal by the second case of equal triangles. If from the whole quadrilateral CABF, we take

the triangle DBF, there will remain the parallelogram ABDC; and if from the same quadrilateral we take the equal triangle CAE, there will remain the parallelogram ABFE; the two parallelograms must therefore be equivalent. We therefore say—*Parallelograms of equal bases and equal heights, are equivalent.*

152. As every rectangle is a parallelogram, it follows that—*Every parallelogram is equivalent to a rectangle of an equal base and equal height.*

153. We have seen that the diagonal of a parallelogram divides it into two equal triangles (60); the trian-
Fig. 83. gle ABC (*fig.* 83) is then one half of the parallelogram ABCD; and therefore (152)—*Every triangle is equivalent to one half of a rectangle of an equal base and equal height.* And consequently—*All triangles of equal bases and equal heights, are equivalent.*

154. We see that the triangle and parallelogram are both referred to the *rectangle* for the measure of their *area.* If then we can find the exact measure of the rectangle, we have the means of measuring the area of any parallelogram or triangle, and, hence, of any rectilinear figure, as all rectilinear figures may be divided into triangles by drawing diagonals through opposite vertices.

155. To compare two rectangles of the same height and unequal bases, place the less upon the greater so that two equal sides may coincide, and the base CF lie
Fig. 84. along the base CD (*fig.* 84).

If the bases are commensurable, divide them by a common measure; and through the points of division, draw lines parallel to the side CA. Suppose this common measure to be contained eight times in AB, and five times in AE; these lines parallel to CA will divide the larger rectangle into *eight* equal rectangles, of which the smaller rectangle AECF will contain five. This will give us the proportion $\frac{ABCD}{AECF} = \frac{AB}{AE}$; that is, the two rectangles are to each other in the ratio of their bases.

Fig. 85. If the bases AB, AE are incommensurable, (*fig.* 85) divide the larger AB by a measure of the smaller. Suppose it to be contained a certain number of times with the indefinitely small remainder B $b$, less than any assignable magnitude, which denote by $d$; and draw through $b$ a straight line parallel to BD. Let us denote the difference between the rectangle whose base is AB and the rectangle

whose base is A$b$ by $m$; as A$b$ and AE are commensurable, we shall have the proportion $\frac{\text{ABCD} - m}{\text{AECF}} = \frac{AB - d}{\text{AE}}$. If we apply the measure of AE one time more than it is contained in AB, it will reach beyond B to $b'$; and suppose this excess B$b'$, which we denote by $d'$, to be less than any assignable magnitude, (since we can divide AE into parts so small that this excess shall be as little as we please), if we draw through $b'$ a line parallel to BD, meeting CD produced, we shall have a rectangle which exceeds the rectangle ABCD by an indefinitely small excess which we designate by $m'$. According to what is proved in the preceding part of this article, we have $\frac{\text{ABCD} + m'}{\text{AECF}} = \frac{\text{AB} + d'}{\text{AE}}$. We see, therefore, that in comparing these two rectangles and their bases, if any magnitude however small be added to the rectangle ABCD, a corresponding magnitude must be added to its base AB to preserve the proportion; and if any magnitude however small be subtracted from the rectangle ABCD, a corresponding magnitude must be subtracted from its base to preserve the proportion; it therefore follows necessarily, that if nothing be added to, or subtracted from the rectangle, nothing must be added to, or subtracted from its base, to preserve the proportion; and we consequently have $\frac{\text{ABCD}}{\text{AECF}} = \frac{\text{AB}}{\text{AF}}$; that is, the rectangles are in the ratio of their bases though the bases are incommensurable. We therefore say—*Two rectangles of the same height, are to each other in the ratio of their bases.*

156. As the side AC might have been taken as the common base of the two rectangles whose heights are AB and AE; we have from the same proportion $\frac{\text{ABCD}}{\text{AECF}} = \frac{\text{AB}}{\text{AF}}$, this general rule also—*Two rectangles of equal bases are to each other in the ratio of their heights.*

157. As a triangle is equivalent to half the rectangle of the same base and height, it follows from the last two articles, that—*Two triangles, of equal heights, are to*

*each other in the ratio of their bases; and two triangles of equal bases, are to each other in the ratio of their heights.*

Fig. 86. 158. If we place two rectangles (*fig.* 86) so as to have the point C a vertex in each, and their sides parallel; and produce BA and FE to meet in H; we shall have another rectangle ACEH, with which each of the others may be compared; these will give the following proportions (155) $\frac{ABCD}{ACEH} = \frac{CD}{CE}$, $\frac{ACEH}{CEFG} = \frac{AC}{CG}$. If we multiply these ratios *in order*, it will give $\frac{ABCD \times ACEH}{ACEH \times CEFG} = \frac{CD \times AC}{CE \times CG}$; but in the first ratio of the proportion, the same factor ACEH, occurs both in the numerator and denominator; it may therefore be stricken out, which will give $\frac{ABCD}{CEFG} = \frac{CD \times AC}{CE \times CG}$. But $CD \times AC$ is the product of the base of the rectangle ABCD by its height, and $CE \times CG$ is the product of the base of the rectangle CEFG by its height. Therefore—*Two rectangles are to each other as the products of their bases by their heights.*

159. It is usual to estimate areas by square feet, square yards, square rods, &c. By a *square foot*, we mean a square whose side is one foot; by a *square yard*, a square whose side is one yard, &c. If we have a rectangular floor whose length is six yards, and breadth five yards, in order to ascertain its area let us divide the end
Fig. 87. of the room AC (*fig.* 87) into yards, and through each division draw lines parallel to AB; this will divide the floor into five equal rectangles, each a yard in width and six yards long; if then we divide the line AB into portions of one yard each, and through these points of division draw lines parallel to A *a*, we shall divide the rectangle AB *a b* into *six* equal squares whose side is one yard; each of the other rectangles *a b c d*, &c. will contain six square yards; so that the whole rectangle will contain five times six square yards, or thirty square yards. This might have been at once obtained by multiplying the number expressing the linear yards in AB, by the number of linear yards in AC.

If we would have the area of a rectangle in square

feet, we must multiply the number of linear feet in one side by the number of feet in the contiguous side. This shows us what is meant by the product of two lines; and we therefore say—*The area of a rectangle is expressed by the product of its base multiplied by its height.*

160. As any parallelogram is equivalent to a rectangle of the same base and height—*The area of any parallelogram is the product of the base multiplied by the height.* And therefore (153)—*The area of a triangle is half the product of the base multiplied by the height. Consequently any two triangles are to each other as the products of their bases by their heights.*

161. If we designate the height of a rectangle by $a$, and the base by $b$, the area will be expressed by $a \times b$; or $a \,.\, b$; and the rectangle is said to be *contained* by the two lines $a$ and $b$. If the base of the rectangle is equal to its height, the figure is a square (62), and its area is expressed by $a \,.\, a$, or $a^2$ (73). Hence the second power of a quantity is frequently called its *square.** A triangle, whose height is $a$, and whose base is $b$, will be represented by $\frac{1}{2} a \,.\, b$, or $\frac{a \,.\, b}{2}$.

162. *Problem.* To construct upon a given line $c$ (*fig.* 88) a rectangle equivalent to the given rectangle $a \,.\, b$. Find a fourth proportional (97) to the three given lines $c, a, b$; let $d$ be this fourth proportional; we have $\frac{c}{a} = \frac{b}{d}$. Multiply both sides by $a$ and also by $d$, we have $c \times d = a \times b$; the rectangle contained by the two lines $c$ and $d$ is the rectangle required. Fig. 88.

163. *Problem.* To find a square equivalent to a given rectangle. Let $a$ and $b$ be the two sides of the given rectangle; find a mean proportional between these two lines (98); let $c$ be this mean proportional; we have $\frac{a}{c} = \frac{c}{b}$; multiply both by $c$ and $b$, we have $a \times b = c^2$; $c$ is the side of the required square.

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* The expression "$a$ square," or "the square of $a$," may be used when, as in the present case, we wish to designate the area of a rectangle, each of whose sides is expressed by $a$; but when the product $a \times a$, or $a^2$, is not considered as representing such an area, it should never be read "$a$ square."

164. *Problem.* To find a square equivalent to a given triangle. Let $a$ be the height of the triangle and $b$ the base. Find $c$ a mean proportional between $a$ and $\frac{1}{2}b$. We shall then have $\frac{a}{c} = \frac{c}{\frac{1}{2}b}$; and by multiplying by both the denominators we obtain $\frac{1}{2}a \times b = c^2$; that is, $c$ is the side of the square sought.

165. Any polygon may be changed into an equivalent polygon of a less number of sides, by using the principle that triangles of the same base and height are equivalent. Suppose we have the pentagon ABCDE. Draw the di-
Fig. 89. agonal AD (*fig.* 89); through the vertex E draw EF parallel to this diagonal and meeting CD produced in F; and draw AF; the pentagon ABCDE will be transformed into the equivalent quadrilateral ABCF. In the same manner all polygons may be reduced to equivalent triangles.

166. If we would ascertain the measure of a trapezoid,
Fig. 90. as ABCD (*fig.* 90), AB and CD being the parallel sides; draw the diagonal AD; this will divide the trapezoid into two triangles whose bases are respectively the two parallel sides of the figure, and whose common height is the perpendicular distance AF of these two sides. Each triangle will have for its measure the height multiplied by half the base (160). Therefore the sum of the two triangles will have for its measure the product of the common height into half the sum of their bases. We therefore say—*The measure of a trapezoid is the product of half the sum of the parallel sides, multiplied by their perpendicular distance.*

If we take $a$ for the height of a *trapezoid*, and $b$ and $b'$ for the two parallel sides, the area will be expressed by $\frac{1}{2}a.(b+b')$, or $\frac{a.(b+b')}{2}$.

167. Let us now compare two *similar triangles*, as
Fig. 91. $M$ and $M'$, (*fig.* 91). Let their heights be $a$ and $a'$, and their bases $b$ and $b'$; this will give $\frac{M}{M'} = \frac{\frac{1}{2}(a.b)}{\frac{1}{2}(a'.b')}$; or leaving out the common factor $\frac{1}{2}$, we have $\frac{M}{M'} = \frac{a.b}{a'.b'}$. But because the triangles are similar, we have (91) $\frac{a}{a'} = \frac{b}{b'}$; we might therefore substitute $\frac{b}{b'}$ for its equal

$\frac{a}{a'}$ in the preceding equation; which gives $\frac{M}{M'} = \frac{b.b}{b'.b'}$; or which is the same thing, $\frac{M}{M'} = \frac{b^2}{b'^2}$. We see, therefore, that the two triangles are to each other in the ratio of *the second powers of their bases.* Any other homologous sides being taken for bases, would give a similar result. And as all homologous lines are proportional in *similar* triangles, we have this general rule—*Similar triangles are to each other as the second powers of their homologous dimensions; or, as the squares described upon their homologous lines.*

168. As two similar polygons of any number of sides (93) are composed of the same number of similar triangles of proportional dimensions, this result may be readily applied to them.

The two similar *pentagons* (*fig.* 92) are divided by the diagonals, each into three triangles similar to the corresponding triangles in the other (93). The homologous diagonals $a$, $b$, and $a'$, $b'$, give the proportion $\frac{a}{a'} = \frac{b}{b'}$ (92). Fig. 92.

The similar triangles, in the two figures, give the proportions $\frac{L}{L'} = \frac{a^2}{a'^2}$, $\frac{M}{M'} = \frac{a^2}{a'^2}$ and $\frac{N}{N'} = \frac{b^2}{b'^2}$, or $\frac{N}{N'} = \frac{a^2}{a'^2}$ (167); therefore (84) $\frac{L+M+N}{L'+M'+N'} = \frac{a^2}{a'^2}$; that is, —*The areas of similar polygons are to each other as the second powers of their homologous dimensions;* or *as the squares described upon their corresponding lines.*

169. Suppose we have the two triangles ABC, AEF, (*fig.* 93) having the common angle A; draw the perpendiculars BD and EG to the side AC; the similar triangles ABD, AEG, give the proportion $\frac{\text{BD}}{\text{EG}} = \frac{\text{AB}}{\text{AE}}$. But the two triangles ABC, AEF, give $\frac{\text{ABC}}{\text{AEF}} = \frac{\text{AC} \times \text{BD}}{\text{AF} \times \text{EG}}$ (160). If we multiply these two proportions in order, and suppress the common factor $\frac{\text{BD}}{\text{EG}}$, we shall have $\frac{\text{ABC}}{\text{AEF}} = \frac{\text{AB} \times \text{AC}}{\text{AE} \times \text{AF}}$. Fig. 93.

That is—*If two triangles have an angle of the one equal*

*to an angle of the other, their areas will be as the products of the sides containing the equal angles.*

Fig. 94. 170. If we take CE (*fig.* 94) equal to the sum of the two lines $a$ and $b$, and construct upon it the square CEFG ; and upon the part CD, equal to $a$, construct the square CDIH, and produce the sides HI and DI to L and K ; we shall have the square of CE (the sum of the two lines $a$ and $b$) divided into *four parts*, viz. CDIH, IKGL, DELI, and HFKI. Because CEGF is a square, CF is equal to CE ; and because CDIH is a square, CH is equal to CD ; therefore FH is equal to DE, or $b$; and HI being one side of the square CD or $a$, the rectangle HFKI is a rectangle contained by two lines, one equal to $a$, and the other equal to $b$; DELI is a rectangle of the same dimensions; and ILGK is a square whose side is equal to DE, or $b$; so that the whole figure, which is the square of CE (the sum of the two lines, $a$ and $b$), contains two squares, which are the squares of the parts $a$ and $b$; and two equal rectangles each of which is contained by these two parts. This gives the equation $(a+b)^2 = a^2 + b^2 + 2(a.b)$, which may be enunciated as follows—*If a line is equal to the sum of two other lines, the square described upon this line is equal to the sum of the squares of the two lines, plus twice the rectangle contained by the two lines.*

Fig. 95. 171. Take (*fig.* 95) CD the difference between two lines $a$ and $b$; construct upon BD, equal to $a$, the square ABDI ; and upon CD, equal to the difference of the two lines, another square CDEF ; produce EF to G, making FG equal to $b$; and complete the square FGHI, which will be the square of $b$; so that the whole figure is equal to the square of $a$ plus the square of $b$; but ABCK is a rectangle contained by the two lines $a$ and $b$; EGHK is an equal rectangle; if we take away these two rectangles, we shall have left the square CDFE, (the square of $a-b$). This gives the equation $(a-b)^2 = a^2 + b^2 - 2(a.b)$, which gives this general rule—*The square of the difference of two lines, is equal to the square of the greater, plus the square of the less, minus twice the rectangle contained by the two.*

172. Let $a$ represent the hypothenuse of a right-angled triangle whose two sides are $b$ and $c$. From the vertex
Fig. 96. of the right-angle draw AD (*fig.* 96) perpendicular to the hypothenuse. We have (101) the two equations, CD =

$\frac{(CA)^2}{CB} = \frac{b^2}{a}$, and $DB = \frac{(AB)^2}{CB} = \frac{c^2}{a}$. Adding these equations, we have $CD + DB = CB = a = \frac{b^2 + c^2}{a}$; and multiplying both sides by $a$, we have $a^2 = b^2 + c^2$, that is—*The square described upon the hypothenuse of a right-angled triangle, is equivalent to the sum of the squares described upon the other two sides.*

173. We may demonstrate this truth from the areas immediately, without referring the lines to numbers, as in the above proof. Upon the hypothenuse CB (*fig.* 97) construct the square CF; and upon the inside of the other sides CA, AB, construct the two squares CH and AL; from the vertex A of the right-angle draw AD perpendicular to the hypothenuse and produce it to G; and produce ML and HN till they meet the periphery of the square CEFB: the points of meeting will be the vertices F and E; for the triangle LBF has its sides respectively perpendicular to those of the original triangle ABC, its angles are therefore equal; and having LB equal to AB, being a side of the same square, the two triangles are therefore equal by the third case, their hypothenuses are consequently equal, and ML produced must meet BF in the point F; in the same manner we may show that HN produced will fall upon the point E. Fig. 97.

The square AL and the parallelogram AKFB, are upon the same base AB, and have the same height BL, they are therefore equivalent (151); BK and BG are parallelograms upon the same base BF, and have the same height BD, they are therefore equivalent; that is, the square upon the side AB is equivalent to the rectangle BG: in the same manner we may show that the square upon CA is equivalent to the rectangle CG: therefore the sum of the squares $b^2 + c^2$ is equivalent to the sum of the rectangles CG and BG; but these two rectangles taken together make the square of the hypothenuse $a^2$. This gives us the equation $a^2 = b^2 + c^2$, the same result as that obtained by the other process.

174. *Remark.* This is one of the most important propositions in elementary geometry. A method of proving it geometrically was long a *desideratum* with the ancient geometers. It was at length discovered by Pythagoras. It is known by the name of *the Pythagorean proposition;* and is the 47th of the 1st book of Euclid.

Next to this in importance, as a geometrical truth, and nearly connected with it, is the proposition—*Similar figures have their homologous measures proportional.* In the application of the elementary principles, these two propositions are continually used.

175. If $b^2 + c^2 = a^2$, then $b^2 = a^2 - c^2$, and $c^2 = a^2 - b^2$; that is, *The square of one side of a right-angled triangle, is equivalent to the square of the hypothenuse, minus the square of the other side.*

Fig. 98. 176. In figure 98, let $AB = a$, and $CB = b$. Produce AB to E, making $BE = b$, then AE will be equal to $a + b$, and AC will be equal to $a - b$. Upon AB make the square AD, which will be the square of $a$ or $a^2$; and upon AC construct the square AF, which will be the square of $a - b$, that is $(a - b)^2$. If we complete the rectangle AH, it will be the rectangle contained by $a + b$ and $a - b$; and as the part $M'$ is equal to the part $M$ of the figure, instead of the rectangle AH, we may take its equivalent AIKFLB; but this part of the figure is equal to the square IB with the square KL taken from it, that is, equal to $a^2 - b^2$; so that the rectangle AH is equivalent to the difference between the square described upon $a$ and the square described upon $b$, which gives the equation $(a + b) \times (a - b) = a^2 - b^2$. *The rectangle contained by the sum and difference of two lines, is equivalent to the difference of the squares described upon the two lines.*

Fig. 97. 177. The two rectangles CEGD, DBFG, (*fig.* 97) of the same height are in the ratio of their bases CD and DB (155); but they are equivalent respectively to the squares of the two sides, CA and AB, to which they are adjacent; consequently—*The squares of the two sides of the right-angled triangle, are to each other in the ratio of the adjacent segments of the hypothenuse.*

178. Suppose that the angle A, instead of being a right-
Fig. 99. angle, is an obtuse angle (*fig.* 99); produce the side CA till it meet in D, the perpendicular BD, drawn from the opposite vertex; denote the perpendicular by $y$, and the distance AD by $x$. The right-angled triangle ABD (172) gives the equation $c^2 = y^2 + x^2$; and the triangle CBD gives $a^2 = y^2 + (x + b)^2 = y^2 + x^2 + b^2 + 2(b \times x)$ (170). And substituting $c^2$ for its value, we have $a^2 = c^2 + b^2 + 2(b \times x)$. We therefore say—*In an obtuse-triangle, the square of the side opposite the obtuse*

*angle, is equivalent to the sum of the squares of the other two sides, plus twice the rectangle contained by one of those sides and the distance of the vertex of the obtuse angle from the foot of the perpendicular drawn from the opposite vertex to this side produced.*

179. If the angle A be acute, the perpendicular BD (*fig.* 100), will fall within the triangle. Designating AD by $x$, and BD by $y$, we have $(CD)^2 = (b - x)^2 = b^2 + x^2 - 2(b \times x)$, (171); $c^2 = y^2 + x^2$; $a^2 = y^2 + (CD)^2 = y^2 + b^2 + x^2 - 2(b \times x)$; and substituting $c^2$ for its value, we have $a^2 = c^2 + b^2 - 2(b \times x)$; that is —*In an oblique-angled triangle, the square of the side opposite an acute angle, is equivalent to the sum of the squares of the other two sides, minus twice the rectangle contained by one of the sides adjacent to this angle, and the distance of this angle from the foot of the perpendicular drawn to this adjacent side from the opposite vertex.*

180. In figure 99, (170) we have $(CD)^2 = x^2 + b^2 + 2(b \times x)$; and subtracting $x^2$ from each side, we have $(CD)^2 - x^2 = b^2 + 2(b \times x)$. We had also on the same figure, $a^2 = c^2 + b^2 + 2(b \times x)$; subtracting $c^2$ from each side we have $a^2 - c^2 = b^2 + 2(b \times x)$; therefore $a^2 - c^2 = (CD)^2 - x^2$.

In figure 100 (179) we have $(CD)^2 = x^2 + b^2 - 2(b \times x)$; therefore $(CD)^2 - x^2 = b^2 - 2(b \times x)$; but we had in the same article $a^2 = c^2 + b^2 - 2(b \times x)$; which gives $a^2 - c^2 = b^2 - 2(b \times x)$; therefore $a^2 - c^2 = (CD)^2 - x^2$. We therefore say—*If from the vertex of either of the angles of any triangle, a perpendicular be drawn to the opposite side (produced if necessary) the difference of the squares of the sides containing this angle, will be equivalent to the difference of the squares of the two distances of the foot of this perpendicular from the other vertices of the triangle.*

181. If we would find the side of a square equivalent to two given squares; we have only to make a right-angled triangle, the two sides of which are respectively equal to the sides of the given squares; the hypothenuse will be the side of the square sought (172).

If we would find the side of a square equivalent to the sum of several squares, as $a^2 + b^2 + c^2 + d^2$, &c. (*fig.* 101); construct a right-angled triangle ABC, whose two sides are $a$ and $b$, the square of the hypothenuse $(m^2)$ will be equivalent to the two squares $a^2$ and $b^2$; then make Fig.101.

another right-angled triangle with the two sides $m$ and $c$; the square of the hypothenuse $(n^2)$ will be equivalent to the sum of the first three squares; that is, $n^2 = m^2 + c^2$; therefore $n^2 = a^2 + b^2 + c^2$; or $n = (a^2 + b^2 + c^2)^{\frac{1}{2}}$; proceed in the same way with any number of squares.

182. If we would find a square equivalent to the dif-
Fig.102. ference between two given squares, $a^2$ and $b^2$ (*fig.* 102); draw a line AC, equal to the side of the smaller of the two squares, $b^2$; at A draw the indefinite line AB, perpendicular to AC; then from C as a centre with a radius equal to the side of the larger square, draw an arc cutting the perpendicular in the point B; BA will be the side of the square sought; for we have $c^2 = a^2 - b^2$; $c = (a^2 - b^2)^{\frac{1}{2}}$.

183. If from the summit of any triangle, as ACB
Fig.103. (*fig.* 103), we draw a straight line, as CE, to the middle of the base, it will divide the triangle into two triangles, one of which (if the sides CA and CB are unequal) will be an obtuse-angled triangle, and the other an acute-angled triangle. If we draw the perpendicular CD, and designate the line CB by $a$, AC by $b$, AE and EB each by $\frac{1}{2}c$, ED by $x$, and CE by $m$; the triangle ACE will give the equation $b^2 = m^2 + (\frac{1}{2}c)^2 + 2\ (\frac{1}{2}c \times x)$, (178); the other triangle gives the equation $a^2 = m^2 + (\frac{1}{2}c)^2 - 2\ (\frac{1}{2}c \times x)$, (179); adding these two equations, we have $a^2 + b^2 = 2\,m^2 + 2\ (\frac{1}{2}c)^2$; that is—*If a line be drawn from the vertex of any angle of a triangle to the middle of the opposite side, twice the square of this line, added to twice the square of half this opposite side, will be equivalent to the sum of the squares of the two sides which contain this angle.*

Fig.104. 184. If in the parallelogram ACBD (*fig.* 104) we draw the two diagonals, they will bisect each other at E; and according to the last article, the triangle ACB gives

$$b^2 + a^2 = 2\,(\text{AE})^2 + 2\,(\text{CE})^2;$$
$$c^2 + d^2 = 2\,(\text{AE})^2 + 2\,(\text{DE})^2;$$

adding now the two equations, and considering that CE equals DE, we have $a^2 + b^2 + c^2 + d^2 = 4\ (\text{AE})^2 + 4\ (\text{CE})^2$. But $4\ (\text{AE})^2$ is the square of 2AE; that is, the square of AB; and $4\ (\text{CE})^2$ is the square of 2CE, the square of CD. The equation, therefore, becomes $a^2 + b^2 + c^2 + d^2 = (\text{AB})^2 + (\text{CD})^2$. That is—*The sum of the squares of the four sides of a parallelogram,*

*is equivalent to the sum of the squares of the two diagonals.*

185. A regular polygon (*fig.* 76) is composed of a certain number of equal isosceles triangles, whose bases make the perimeter of the polygon, and whose common height C *a* is the apothegm of the polygon. The area of each of these triangles is the product of its base into half the height; therefore—*The area of a regular polygon is expressed by the product of its perimeter multiplied by half its apothegm;* that is, *by half the radius of the inscribed circle.* Fig. 76.

186. As two regular inscribed polygons of the same number of sides are composed each of the same number of equal triangles; and as the triangles in the one polygon are similar to those of the other, these similar triangles (*fig.* 105) will give the proportion $\frac{BOC}{boc} = \frac{(BO)^2}{(bo)^2} =$ Fig.105. $\frac{(ON)^2}{(on)^2}$; or $\frac{6(BOC)}{6(boc)} = \frac{(BO)^2}{(bo)^2} = \frac{(ON)^2}{(on)^2}$. But six times BOC is the larger polygon, and six times *boc* is the smaller polygon; BO, *b o*, are radii of the circumscribed circles; and ON, *o n*, are radii of the inscribed circles. We therefore say—*The areas of regular polygons of the same number of sides, are to each other as the squares of the radii of the circumscribed circles, and also of the inscribed circles.*

187. It is manifest that as we increase the number of sides in the inscribed regular polygon, we increase its area, which may approach indefinitely near to that of the circle, but can never exceed it. And as we increase the number of sides in the regular circumscribed polygon, we diminish its area, which, though it may approach indefinitely near to that of the circle, can never be less. We have also seen (148) that as the number of sides of the inscribed and circumscribed polygons are increased, their perimeters approach in value to that of the circumference.

If we suppose that the number of sides of the circumscribed polygon is so great that the excess of its perimeter over the circumference is less than any assignable magnitude, which excess we denote by *d;* the area of the polygon will exceed the area of the circle, which we express by *A*, by an exceedingly small quantity, which

we denote by $m$. If we denote the circumference by $C$, the perimeter of the polygon will be expressed by $C + d$; the radius of this circle, which we denote by $R$, will evidently be the apothegm of this circumscribed polygon, and its area will be expressed by $A + m = \frac{1}{2}R \times (C + d)$.

The perimeter of the inscribed polygon of the same number of sides will be less than the circumference of the circle by an indefinitely small magnitude $d'$; and the area of this polygon will be less than that of the circle by an indefinitely small magnitude $m'$. The apothegm of this inscribed polygon will be less than the radius of the circle by an indefinitely small magnitude $x$. To express the area of this inscribed polygon, we have the equation $A - m' = \frac{1}{2}(R - x) \times (C - d')$. The area $A + m$ is greater than the area of the circle by an indefinitely small quantity, and the product $\frac{1}{2} R \times (C + d)$ which expresses this area, is greater than $\frac{1}{2} R \times C$. The area $A - m'$, is less than the area of the circle by an indefinitely small quantity, and the product $\frac{1}{2}(R - x) \times (C - d')$ which expresses this area, is less than $\frac{1}{2} R \times C$. If, then, any area *greater* than the area of the circle requires for its expression a product *greater* than $\frac{1}{2} R \times C$; and if any area *less* than the area of the circle requires for its expression a product *less* than $\frac{1}{2} R \times C$; it manifestly follows that the product $\frac{1}{2}R \times C$ cannot be the measure of an area greater or less than that of the circle whose radius is $R$ and circumference $C$; it must therefore be the measure of an area equal to the area of this circle. Hence we say—*The area of a circle has for its measure half the product of radius multiplied by the circumference.* That is—*A circle is equivalent to a rectangle whose base is equivalent to the entire circumference, and whose height is equal to half radius.*

188. Having a rectangle equivalent to a given circle, a mean proportional between its two adjacent sides, will be the side of an equivalent square.

To find the side of a square equivalent to a circle of a given diameter, constitutes the hitherto unsolved problem of *the quadrature of the circle.* This problem, it will be seen, depends upon another, that is, to find the circumference of a circle, the radius or diameter being given, or in other words, to find the definite ratio of the circumference of a circle to its diameter. This is called—*The definite rectification of the circumference of a circle.*

Although the length of the circumference of a given circle, or its ratio to the diameter, has never been exactly obtained; yet it may be approximated to any required degree of exactness.

189. *Problem.* To find the *approximate ratio* of the circumference to the diameter of the circle. Designating this ratio by $\pi$, the radius of any circle by $R$, the circumference of a circle whose radius is $R$, by $C$, and the area of the same circle, by $A$, we have $\pi = \frac{\pi}{1} = \frac{C}{2R}$; whence $C = 2\pi \times R$, and $A = 2\pi \times R \times \frac{1}{2} R = \pi . R^2$.

If $R = 1$, then $A = \pi$; that is—*The area of a circle whose radius is one, equals the value of* $\pi$; that is, the area of the circle whose radius is 1, bears the same ratio to the square whose side is one, as the circumference bears to the diameter. The question is therefore reduced to finding the area of a circle whose radius is unity or *one*. The area of a circle being less than that of any circumscribed polygon and greater than that of any inscribed; and as the values of two similar polygons, the one inscribed and the other circumscribed, approach more nearly to an equality, in proportion as the number of their sides is greater; and as the number of sides of these polygons may be increased till the difference in their areas shall be less than any assignable quantity $d$: If then we take the arithmetical mean between the areas of the circumscribed and inscribed polygons as the area of the circle, the error will be much less than half $d$.

A square circumscribed about a circle whose radius is 1, is the square of the diameter, or the square of 2, equal to 4; the inscribed square (143) is the square of $(2)^{\frac{1}{2}}$, or 2. The proposed approximation, therefore, will be readily made if we can find an easy solution of the following problem.

190. *Problem.* The area of a regular inscribed polygon and that of a similar circumscribed polygon being given, to find the areas of regular inscribed and circumscribed polygons of double the number of sides. Let EF and GH (*fig.* 106) be the sides of regular polygons of $n$ sides, the one circumscribed and the other inscribed; by drawing the chords BG and BH, and the tangents GI and HK, the straight lines BG and IK will be the sides of regular Fig. 106.

polygons of $2n$ sides, the one inscribed and the other circumscribed. The triangles ECF and GCH will be contained $n$ times in the respective circumscribed and inscribed polygous of $n$ sides, of which they are parts; and the triangles ICK and BCG, will be contained $2n$ times, in the regular circumscribed and inscribed polygons of $2n$ sides. Let $A$ represent the area of the regular circumscribed polygon of $n$ sides; $a$ the area of the similar inscribed polygon; $A'$ and $a'$ the areas of the regular circumscribed and inscribed polygons of $2n$ sides. We shall have the areas of the several polygons as follows: circumscribed, of $n$ sides $= A = 2n \times \text{EBC}$,
inscribed, of $n$ sides $= a = 2n \times \text{GDC}$,
circumscribed, of $2n$ sides $= A' = 4n \times \text{BCI}$,
inscribed, of $2n$ sides $= a' = 2n \times \text{GBC}$.

Then $\dfrac{A}{a'} = \dfrac{\text{EBC}}{\text{GBC}}$, and $\dfrac{a'}{a} = \dfrac{\text{GBC}}{\text{GDC}}$; and triangles of the same altitude being to each other as their bases, we have $\dfrac{\text{EBC}}{\text{GBC}} = \dfrac{\text{CE}}{\text{CG}} = \dfrac{\text{CB}}{\text{CD}}$; for the same reason, we have $\dfrac{\text{GBC}}{\text{GDC}} = \dfrac{\text{CB}}{\text{CD}} = \dfrac{\text{EBC}}{\text{GBC}}$; consequently $\dfrac{A}{a'} = \dfrac{a'}{a}$, which gives $a' = (A \times a)^{\frac{1}{2}}$. - - - (1).

Also, because CI bisecting the angle BCE, divides the base into parts BI and IE, proportional to the sides CB and CE, we have

$$\frac{\text{BCI}}{\text{ICE}} = \frac{\text{BI}}{\text{IE}} = \frac{\text{BC}}{\text{EC}} = \frac{a}{a'}, \text{ and } \frac{\text{BCI}}{\text{BCI} + \text{ICE}} = \frac{\text{BCI}}{\text{BCE}} = \frac{a}{a + a'} = \frac{\frac{1}{4}A'}{A} = \frac{A'}{2A};$$

but the equation $\dfrac{A'}{2A} = \dfrac{a}{a + a'}$ gives $A' = \dfrac{2A \times a}{a + a'}$. - - - (2).

The problem then is solved by means of these two formulas; for knowing the areas of the inscribed and circumscribed polygons of $n$ sides, we have the quantities $A$ and $a$; from these, the formulas give us $A'$ and $a'$, the areas of the inscribed and circumscribed polygons of $2n$ sides.

191. We are now prepared to solve the problem enunciated in article 189, viz.—To find the area of a circle whose radius is *one;* or, in other words, to find the value of $\pi$. The area of the circumscribed square, radius being

1, is equal to 4; the area of the inscribed square is equal to 2. Making $A = 4$, and $a = 2$, the formulas (1), (2), will give $a' = (8)^{\frac{1}{2}} = 2{,}8284271$ = the area of an inscribed octagon; and $A' = \frac{16}{2 + 2{,}8284271} = 3{,}3137085$ = the area of a circumscribed octagon.

If we substitute these values of $A'$ and $a'$, for $A$ and $a$ in the formulas, we shall have $a' = 3{,}0614674$ = the area of a regular inscribed polygon of 16 sides, and $A' = 3{,}1825979$ = the area of a regular circumscribed polygon of 16 sides.

The values of these polygons will enable us to find the areas of inscribed and circumscribed polygons of 32 sides; and, as the farther we proceed in the calculation the nearer will the two polygons approach to an equality, we may continue the process till the values of the inscribed and circumscribed polygons do not differ for any number of decimals to which it may be thought best that the expressions should extend. Having carried the process thus far, and knowing that the circle cannot be greater than the circumscribed polygon, nor less than the inscribed polygon, we take this value for the area of the circle, as far as the expression extends.

The following table gives the result of this calculation pursued till the expressed values of the circumscribed and inscribed polygons in a circle whose radius is 1, do not differ for the first seven decimals.

| Number of sides in the polygon. | Value of circumscribed polygon. | Value of inscribed polygon. |
|---|---|---|
| 4 | 4,0000000 | 2,0000000 |
| 8 | 3,3137085 | 2,8284271 |
| 16 | 3,1825979 | 3,0614674 |
| 32 | 3,1517249 | 3,1214451 |
| 64 | 3,1441148 | 3,1365485 |
| 128 | 3,1422236 | 3,1403311 |
| 256 | 3,1417504 | 3,1412772 |
| 512 | 3,1416321 | 3,1415138 |
| 1024 | 3,1416025 | 3,1415729 |
| 2048 | 3,1415951 | 3,1415877 |
| 4096 | 3,1415933 | 3,1415914 |
| 8192 | 3,1415928 | 3,1415923 |
| 16384 | 3,1415927 | 3,1415925 |
| 32768 | 3,1415926 | 3,1415926 |

We therefore take 3,1415926 for the approximate area of a circle whose radius is 1. It is also the approximate value of the ratio of the circumference of any circle to its diameter, that is, $\pi = 3{,}1415926$ nearly; and multiplying the diameter of a circle by this quantity, will always give us the circumference to a sufficient degree of exactness.

192. If we denote the radius of a circle by $R$, the diameter will be $2R$, and the circumference $2\pi.R$, and the area, $A = \frac{1}{2}R \times 2\pi.R = \pi.R^2$. That is—*The square of radius multiplied by* 3,1415926, *will give the area of any circle.*

193. Let $A$ denote the area of a circle whose radius is $R$, and $a$ the area of a circle whose radius is $r$; we have the two equations, $A = \pi.R^2$, $a = \pi.r^2$. These two equations will give the proportion $\frac{A}{a} = \frac{\pi.R^2}{\pi.r^2}$; and dividing each term of the second ratio by the common factor $\pi$, we have $\frac{A}{a} = \frac{R^2}{r^2}$; that is—*The areas of circles are to each other as the squares of their radii.* The radii of circles are in the ratio of their diameters, and therefore in the ratio of their circumferences (148). We see, therefore, that it is with curvilinear as with rectilinear figures (168); and we have this general rule—*Similar plane figures are to each other as the squares of their homologous lines.*

Fig.107. 194. The sector AEBC (*fig.* 107) is evidently the same part of the circle that the arc AEB is of the entire circumference; if the circumference multiplied by half radius gives the area of the entire circle, it follows that —*The arc of the sector multiplied by half radius, gives the area of the sector.*

To obtain the area of the *segment* AEBD, subtract the area of the triangle ABC from the area of the sector.

PROBLEMS.

(1). Suppose a castle, whose walls are 48 feet high, surrounded by a ditch 64 feet wide; what is the length of a ladder which will reach from the outside of the ditch to the top of the castle wall?

(2). A triangular field has one right-angle; the side opposite the right-angle measures 75 chains; one of the

sides adjacent to the right-angle, measures 45 chains; what is the length of the third side?

(3). How many square chains in a rectangular field whose length is $12\frac{1}{2}$ chains, and whose width is 8 chains?

(4). What is the side of a square field of an equal area with the above?

(5). One side of a triangular field is 20 rods; and the perpendicular distance of the vertex of the opposite angle from this side, is 20 rods; what is the side of a square field whose area is twice as great?

(6). What length of carpeting $\frac{7}{8}$ of a yard wide, will cover the floor of a room whose length is 30 feet and width 21 feet?

(7). The length of a rectangular lawn is 25 rods, and its width 8 rods; what is the length of an equivalent lawn whose width is $12\frac{1}{2}$ rods?

(8). What is the circumference of a circle whose radius is 4? and what is the area of the same circle?

(9). What is the approximate value of the side of a square equivalent to a circle whose radius is 9?

(10). What is the radius of a circle whose circumference is 31,416?

(11). What is the value of the apothegm of a regular hexagon, inscribed in a circle whose radius is 20? and what is the area of this hexagon?

(12). What is the area of an equilateral triangle inscribed in a circle whose radius is 20?

(13). What is the area of a square inscribed in a circle whose radius is 10? What is the area of the circumscribed square?

(14). What is the area of a semi-circular pond whose straight side is 200 yards?

(15). What is the area of a sector embracing 60° in a circle whose radius is 20 yards?

# PART SECOND.

## SECTION I.—*Of Planes, and of Bodies terminated by Planes.*

195. IN the preceding sections, we have considered all the lines of any one of the figures which have been discussed, as in a plane surface; that is, in a surface to which a straight line may be applied in every direction, so as to lie entirely in this surface. This has enabled us to unfold the elementary principles of linear geometry, and to measure the areas of surfaces bounded by straight lines or circular curves. We proceed now to investigate the geometrical relations in *bodies.*

As the surfaces of many bodies consist, in whole or in part, of planes; and as all bodies involve the *three dimensions* of space, length, breadth and thickness; in discussing their geometrical character, it is necessary to consider not only the forms and magnitudes of planes, but their relative positions, their inclinations, and their relation to lines and points *without* them.*

In discussing the general relations of lines to planes, and of planes to each other, we consider them as indefinitely extended, except where some limit is expressly stated.

196. If two straight lines are in the same plane, a second plane may be conceived to pass through one of these lines without coinciding with the other line; it will therefore cut the first plane. But if any two points

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* In what follows, the figures though represented on a plane, are considered as embracing the three dimension of space. And where any line is supposed to pass behind any part of a body or of a plane, it is indicated in the figure by a dotted line.

common to the two planes, be taken, a straight line drawn between them will lie wholly in each of the planes (8); therefore—*The intersection of two planes is a straight line.*

197. If the second plane be turned about the straight line through which it passes, till it coincides with the other line, it must also coincide with the other plane: *Two straight lines, therefore, are sufficient to determine the position of a plane: And three points, not in the same straight line, will also determine the position of a plane;* for the plane may be conceived to revolve about a straight line passing through two of the points; and if it be placed so as to coincide with the other point, it can revolve no farther without leaving it; it is therefore fixed.

It follows from this that any two straight lines which cut each other, are in the same plane; for a plane may be passed through one of them, as AB (*fig.* 108), and turned till it coincide with the other CD. Fig.108.

It is evident, therefore, that—*any three points*, not in the same straight line, being joined, two and two, by three straight lines, *a plane triangle* will be formed; if *four* points are so joined, the quadrilateral, thus formed, *may* be a plane figure; but if the plane passing through three of the points, do not at the same time pass through the *fourth*, the quadrilateral cannot be considered as a single surface.

198. *A straight line* is said to be *parallel to a plane, when it does not incline towards the plane, in either direction.* A line parallel to a plane will therefore be, in every part, *at the same distance from the plane;* and consequently *can never meet it, however far produced.*

*A straight line is perpendicular to a plane* when its inclination to the plane is the same on all sides. It is evident that the straight line will, in this case, be perpendicular to every straight line drawn in this plane, through the *foot of the perpendicular.*

199. Let us suppose a straight line PD (*fig.* 109) to revolve about the straight line AB, always perpendicular to it, AB being considered as fixed. It is evident that PD, by this revolution, will *generate* a plane surface; for in whatever position the revolving line be placed, in the position PE, for instance, a plane may be conceived to pass through EPA, which being extended Fig.109.

Fig.109. will also pass through the opposite position of the revolving line, as PC. But as this line must be always perpendicular to AP, the sum of the two angles APE, APC, will be equal to two right-angles, and CPF will always be a straight line, that is, the surface will be *a plane*, to which AB will be perpendicular (8).

It follows, therefore, that—*If two straight lines are perpendicular to the same straight line, at the same point, they are in the same plane perpendicular to this line.*

200. *Remark.* It is evident that—*There can be only one line passing through* P, *perpendicular to this plane;* for if there could be another it must diverge from PA, and therefore its inclination to the plane must be greater on one side than the other, which cannot be the case with a perpendicular (198).

201. It is also evident that—*Through any point, without a plane,* as A for instance above the plane MN, *only one perpendicular can be drawn to that plane.* For a straight line passing through A in the same direction with AP must be the same line; another line, therefore, which passes through A must be inclined to AP, and consequently will be differently inclined to the plane MN; it cannot, therefore, be perpendicular to this plane.

It is equally evident that—*Only one plane can cut a straight line at the same point, perpendicular to it.* For two planes passing through the same point, must be inclined to each other; they cannot, therefore, be both perpendicular to the same straight line.

202. As any two of the lines, CE, FD, passing through the point P in this plane, are sufficient to fix the position of this plane with respect to the line AB perpendicular to these two lines (197); we say—*If a straight line is perpendicular to each of two straight lines drawn in a plane through its foot, it is perpendicular to every other straight line drawn through its foot in this plane, and is therefore perpendicular to the plane.*

203. Suppose AP to be perpendicular to the plane
Fig.110. MN (*fig.* 110); draw in this plane, from the foot of this perpendicular, the equal straight lines PC, PD; and draw AC, AD. The two triangles APC, APD, are both right-angled at P; the sides containing the equal angles are equal respectively; the triangles are therefore equal by the second case, and give AC = AD. We therefore

say—*Oblique lines drawn to a plane from any point without it and equally distant from a perpendicular to the plane, drawn through the same point, are equal.*

204. It follows from the last result, that—*If from any point without a plane, a straight line be drawn perpendicular to that plane, and also several equal oblique lines be drawn to the plane from this point; these oblique lines will meet the plane in the circumference of a circle whose centre is the foot of the perpendicular.* Consequently—*If a straight line be drawn perpendicular to a circle through its centre, any point in this straight line is equally distant from every point in the circumference of the circle.*

205. If we produce PD to G, and draw AG, it will be easy to show that AG must be greater than AD (51), and thence to show that—*Of two oblique lines falling upon a plane at unequal distances from a perpendicular, that is the greater which falls at the greater distance from the perpendicular.*

206. Suppose the line AG (*fig.* 111) to be oblique to the plane MN; draw from A to the plane the perpendicular AP; join PG, and through G draw BC perpendicular to PG. Then taking GC equal to GB, draw PB, PC, AB, AC; PB and PC will be equal (41), and AB and AC will therefore be equal; and AG will be a straight line drawn from the vertex of an isosceles triangle to the middle of the base; that is—BC *is perpendicular to the oblique line* AG, *when it is perpendicular to the straight line which joins the foot of the oblique line with the foot of the perpendicular* AP. Fig.111.

In this case AP and BC are said to be perpendicular to each other, though they cannot meet.

207. Suppose the plane OP (*fig.* 112) to cut the plane MN, in the line AB; draw through the point C, CD in the plane MN, and CE in the plane OP, each perpendicular to the intersection AB of these planes; then turning the plane OP about the intersection till the two planes coincide, CE will coincide with CD. If the part E of the plane OP be now raised from the plane MN, turning this plane about the *intersection*, this point will describe the arc DE, which in every stage of the process will measure the inclination of the two planes; the centre of this arc is C; therefore—*To measure the angle which two planes make with each other, draw in each plane, through the same point, a straight line perpendicular to the intersec-* Fig.112

*tion, and the angle which these two lines make with each other is the angle of the planes.*

208. *Remark.* The angle made by two planes is called a *diedral angle*, that is, an angle of *two faces;* and is designated by four letters, of which the two middle ones are in the intersection, and the two others are in the different planes out of the intersection; as the diedral angle PABN. The intersection AB of the two planes is called the *edge* of the diedral angle. When this angle is 90°, the inclination of the plane PO to the plane MN, is the same on each side of the intersection, and the planes are said to be *perpendicular* to each other.

It is evident that two diedral angles are to each other as the arcs of the plane angles which are their measures. It is also manifest that two diedral angles which are *opposite at the edge*, as MABO and PABN, are equal.

Fig.113. 209. Suppose the two planes CD and EF (*fig.* 113) to be each perpendicular to the plane AB; from the point G where the three planes meet, draw EH perpendicular to the plane AB; it will be in each of the planes CD and EF (207); therefore—*The intersection of two planes each perpendicular to a third plane, is also perpendicular to this third plane.*

210. If two straight lines are perpendicular to the same plane, they have the same direction in space and are therefore parallel to each other. Let CE be perpen-
Fig.114. dicular to the plane MN (*fig.* 114), and BD be parallel to CE; let the plane AD pass through the parallels; this plane will be perpendicular to the plane MN (207), and DB parallel to EC, will be perpendicular to the intersection AB; it will therefore have the same inclination to the plane MN, as the plane AD has to the plane MN; and as this inclination is a right angle, DB is perpendicular to the plane MN. Therefore—*A straight line parallel to a second which is perpendicular to any plane, is also perpendicular to this plane.*

Fig.115. 211. Take the two planes AB and CD (*fig.* 115) perpendicular to the same straight line GH; draw GR and GL in the plane AB, and from the point H draw HM and HS parallel respectively to GL and GR; GL and GR will be both perpendicular to the line GH, and consequently will lie in the plane AB (201). Now the two lines HM and HS determine the position of the plane CD (198), and GL and GR determine the position of the

plane AB; but HM and GL are parallel, therefore the planes in which they are situated are not inclined to each other in that direction; for the same reason they are not inclined to each other in the direction of the parallels HS and GR. The same may be shown with respect to lines in the two planes in every direction from the points G and H; consequently the two planes are not inclined to each other in any direction; and we say that—*Two planes are parallel when they are not inclined to each other, in any direction.* We say, therefore, that —*Two planes perpendiculor to the same straight line, are parallel.* And—*Two parallel planes have common perpendiculars.*

212. *Remark* 1st. As two parallel planes are not inclined to each other, thy must be *throughout at the same distance from each other;* and, *however far produced, can never meet;* therefore—*All the perpendiculars between parallel planes, are equal.*

213. *Remark 2d.* If two straight lines which cut each other, are parallel respectively to two other straight lines which cut each other, the plane determined by the first two lines, will be parallel to the plane determined by the other two (211).

214. Let the two parallel planes AB and CD (*fig.* 116) be cut by a third plane FH. The two intersec- Fig.116. tions EF and GH, being in the same plane FH, will meet unless they are parallel (12); and if these lines will meet, the planes AB and CD in which they are situated, will also meet; but these planes being parallel to each other, can never meet; consequently the two intersections EF and GH cannot meet; they are then parallel; and we say—*When two parallel planes are cut by a third plane, the two intersections are parallel.*

*Remark.* If the lines FG and EH are parallel, being between the parallels EF and GH they must be equal (60); we say therefore—*Parallel lines between parallel planes, are equal.*

215. Let the two straight lines GH and IK (*fig.* 117) be cut by the three parallel planes, AB, CD, EF. Fig.117. Draw HI piercing the plane CD in M; draw also GI and LM, MN and HK. Then GI and LM are the intersections of the plane HGI with the parallel planes AB and CD, they are therefore parallel; for a similar reason MN is parallel to HK. In the triangle GHI, LM being par-

allel to the base GI, we have $\frac{GL}{LH} = \frac{IM}{MH}$; and in the triangle IHK, MN being parallel to HK, we have $\frac{IM}{MH} = \frac{IN}{NK}$; but these two equations having the common member $\frac{IM}{MH}$, the other two members are equal; that is, $\frac{GL}{LH} = \frac{IN}{NK}$; and we say—*If two straight lines meet three parallel planes, they will be cut into proportional parts by these planes.*

216. When several planes, as ASB, BSC, CSD, DSE,
Fig 118. ESF, FSG, and GSA, (*fig.* 118) all passing through the point S, are joined, two and two, by their sides diverging from S, the space which they comprehend, unlimited in the direction opposite to S, is called a *polyedral angle.** Angles of this kind are distinguished by the number of planes which meet at the summit or vertex of the angle. A *triedral angle* is an angle of *three faces*, as the angle SABC (*fig.* 119); a *tetraedral angle* is one which has *four faces.* The angle SABCDEFG (*fig.* 118) has *seven* faces and is called a *heptaedral angle*, &c.

The things to be considered in a polyedral angle, are the plane angles formed by the edges of the polyedral angle; and the respective inclinations of these planes, or the diedral angles formed by contiguous faces.

217. *Problem. Three plain angles being given to construct a triedral angle.* Upon the side AS of one of the
Fig.119. angles, ASB (*fig.* 119), place the side AS of another of them, so that their vertices may coincide at S; then raising the part C of the second, make it turn about the intersection AS; it will be perceived that, as the inclination of the plane CAS to the plane BAS increases, the angular space CSB will increase; there is therefore a certain inclination of the first two planes, which allows the third angle to be introduced to complete the triedral angle; therefore the magnitude of each of the plane angles, determines the inclination of the other two. We say therefore—*If two triedral angles are formed of plane angles respectively equal, the inclination of any two planes in the one, will be equal to the inclination of the homologous planes in the other.*

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* This was formerly called a *solid* angle.

218. *Remark* 1. If the plane angles be unequal, but are placed in the same order in both, the two triedral angles will coincide, when placed together; in this case they are called *equal*. If the order of the planes be different (*fig*. 120), the two triedral angles cannot coincide; for, placing two equal faces together, so that they may coincide, two other equal faces will be inclined to this plane on opposite sides. In this case they are said to be *symmetrical*. Fig.120.

219. *Remark* 2*d*. If the three plane angles which form a triedral angle are all equal, the three inclinations, that is, the three diedral angles will be equal. If each of the plane angles is a right-angle, the diedral angles will be right-angles; and the triedral angle may be called *right-angled*.

220. If the magnitude of the third angle is just equal to the difference between the other two, this third plane can be introduced without inclining the other two, that is, the diedral angle will be nothing; the three planes will then be in the same plane, and will therefore comprehend no space. If the third angle be less than the difference between the other two, it will not fill the remaining space, the inclination of the two being nothing. We say, therefore, that—*The sum of any two of the plane angles which form a triedral angle, is always greater than the third.*

221. *Remark*. Though the magnitude of the plane angles determines their inclination, in a triedral angle; this is not the case with any other polyedral angle. If we have *four* plane angles given, we can form an infinite variety of *tetraedral angles* with them, as they will admit of an infinite variety of inclinations; but if, in a tetraedral angle, *one* of the inclinations be fixed (the plane angles and their order of arrangement being given) this fact will determine all the others.

222. If the sum of the three plane angles given to form a triedral angle (*fig*. 121) were equal to four right angles, the angular space left for the introduction of the third plane, would not be sufficient to receive it until the second plane was turned quite over into the same plane with the first (15). In this case the three plane angles would be in the same plane. The same would be true, of any number of plane angles given to form a polyedral angle; if their sum were equal to four right-angles Fig.121.

their sides would coincide when they were all in the same plane, their vertices being placed together. And it is evident that, if the sum of three angles were *greater* than
Fig.122. four right angles (*fig.* 122) they could not be brought together at their vertices, their sides coinciding two and two; they could not therefore form a triedral angle. Whence we say—*The sum of all the plane angles which form a convex polyedral angle, is always less than four right-angles.*

223. *Remark.* By a *convex* polyedral angle, we mean one, all of whose diedral angles are *salient*, that is, have their edges *standing out*, as in figure 118. When there are, in the polyedral angles, *re-entering* diedral angles,
Fig.123. (as the diedral angle DESF (*fig.* 123), there is no general limit to the sum of the plane angles which may form a polyedral angle.

It is evident that the sum of the diedral angles in triedral angles, varies with the plane angles of which the triedral angle is composed. Are there any limits to this variety? and if so, what are these limits?

### *Of Polyedral Bodies.*

224. From the discussion of polyedral angles, we proceed naturally to that of *polyedral bodies*, or *polyedrons*,* that is—bodies whose surfaces are composed entirely of planes. The word *polyedron* signifies *a body of many faces.* When we speak of *geometrical bodies* we do not include necessarily the idea of *matter* or *substance* which resists the approach of other bodies or excludes them from the places which these occupy, as a block of wood or of marble. Geometry, as we said at first, is the science of *magnitude* and *form*, and takes no cognizance of even the existence of matter. It discusses, it is true, the form and extent of material substances; but it is the *form and extent merely;* or rather, it is the form and meas-

* These bodies have been called *solid polyedrons;* but it has been very properly objected to the term *solid*, that it has in common language an appropriate use, totally different from that which has been assigned to it in geometry. This new application of a familiar term, is calculated to mislead the learner, and should never be adopted.

ure of those portions of space in which these bodies are extended, that makes the subject of geometry. When we speak of a geometrical body, therefore, we mean a portion of space of that particular form and extent; and when we speak of the face of a polyedral body, we mean that *ideal division* between the portion of space occupied by the body, and the portion without.

225. In a polyedral angle it is observed, that the figure is open and unlimited in the direction opposite to the summit (216); it follows, therefore, that with only *one* polyedral angle, of how many planes soever it be composed, no portion of space can be entirely enclosed. In order then to form a *polyedral body*, it will be necessary to have *more than one* polyedral angle.

The simplest polyedral angle we have seen (*fig.* 124) Fig.124.
is one of three faces (216). If these three planes were cut by a fourth, a certain portion of space would be enclosed; and if the contiguous sides of the plane faces, forming a triedral angle, are equal, the sides opposite to the common vertex S in these triangles, as AB, BC, and CA, will be in the same plane (197); the plane triangle ABC may therefore be considered as one face of the polyedron SABC. Without this last plane ABC, we had but one polyedral angle, that whose vertex is at S; we have now *four* polyedral angles; the summits of which are S, A, B and C. Each of these is a triedral angle, the most simple of polyedral angles; and each of the four faces is a triangle, the simplest of rectilinear figures. It seems quite evident, therefore, that if merely these circumstances are considered, this is the *simplest* of polyedral bodies. It is called *a tetraedron*, a body having *four faces.*

226. We consider this body as contained by the three triangles ASB, BSC, CSA, (having a common vertex in S,) and the triangle ABC; which last we will call the *base* of the polyedron.

If we had four triangles, having the vertices in S (*fig.*
125) and their bases in the same plane ABCD, the body Fig.125.
contained by these *five* planes we should call a *pentaedron*, and in general, a body contained by *six* plane faces is called a *hexaedron*; *of eight, an octaedron*, &c. But all bodies of which one of the faces is *a polygon*, and all the others triangles having their summit in the same point, are called *pyramids.* The body SABCDE (*fig.* 126) is Fig.126.

a *pentagonal* pyramid, because its *base* ABCDE is a pentagon: the point S is called the *summit* of the pyramid.
Fig.127. The tetraedron SABC (*fig.* 127) is a *triangular* pyramid. And in general, a pyramid is called *triangular*, *quadrangular*, *pentagonal*, *hexagonal*, &c., according as its base is a *triangle*, *quadrangle*, *pentagon*, *hexagon*, &c.

In any pyramid, each of the polyedral angles adjacent to the base, has only *three* faces. In a tetraedron, therefore, we may take either of the faces for the base; but in a pyramid of a greater number of faces, we can never take a triangle for the base.

227. If the three faces which form the tetraedral an-
Fig.124. gle whose summit is at S (*fig.* 124) were equal issoceles triangles, their respective inclinations to each other would be the same (217); and they would have the same inclination to the base ABC, which must be an equilateral triangle as it is contained by the bases of three equal isosceles triangles. The body would then be called a *regular pyramid.* But it would not be a regular tetraedron, or a regular polyedron. To be a *regular tetraedron*, *all* the faces must be equal. They will, therefore, be equilateral and equally inclined to each other, and the triedral angles will be all equal. And in general, *a regular polyedron* has all its faces equal regular polygons, all its diedral angles equal, and all its polyedral angles equal.

228. Of *a regular pyramid* it is requisite that the base should be a regular polygon of any number of sides, and that the other faces be equal isosceles triangles; these triangles will be therefore equally inclined to the base (217), and a perpendicular drawn from the summit or *apex* (as it is sometimes called) will meet the base in the centre. This perpendicular measures the *height* of the pyramid.

229. We have seen that the plane angles in a tetraedron determine the inclinations of the planes respectively, and consequently the triedral angles; therefore—*If two triangular pyramids are contained by planes respectively equal and similarly arranged, they are equal to each other; and if placed together will coincide in all their parts.*

230. Now let the two triangular pyramids S-ABC,
Fig.128. S-DEF, (*fig.* 128) be contained by similar triangles, arranged in the same order in both. Similar triangles have

their angles equal; therefore the two pyramids will have their triedral angles respectively equal (217), and the inclination of any two faces in the one, equal to that of the corresponding faces in the other. These bodies are therefore called *similar pyramids;* they are also called *similar tetraedrons.* And generally, we call *similar polyedrons*, those which have the same number of faces respectively similar and similarly arranged, and their diedral angles respectively equal. We have seen that, in the tetraedron and the pyramids, this last circumstance is a necessary result of the others (226). We have, therefore, this general truth—*Any two pyramids are similar when the faces of the one are similar respectively to the faces of the other, and similarly disposed.*

231. In the tetraedron SDEF, let us suppose a *section* to be made by passing a plane through ABC, parallel to the base DEF; AB will be parallel to DE, being the intersection of the plane SDE with the two parallel planes *ABC, DEF*, (209); for a similar reason, BC is parallel to EF, and AC to DF. But in any triangle a line drawn parallel to the base, cuts off a partial triangle similar to the whole; therefore the triangle SAB is similar to SDE, SBC similar to SEF, and SAC to SDF. Now in the triangle SDE, AB being parallel to DE, we have $\frac{SE}{SB}=\frac{DE}{AB}$; and BC being parallel to EF, we have $\frac{SE}{SB}=\frac{EF}{BC}$, therefore $\frac{DE}{AB}=\frac{EF}{BC}$; in a similar manner we show that $\frac{DE}{AB}=\frac{DF}{AC}$; that is, the two triangles ABC and DEF have their sides proportional, and are similar, and the tetraedron SABC is similar to the tetraedron SDEF (229). And we say generally—*If a tetraedron be cut by a plane parallel to one of its faces, the portion containing the triedral angle opposite to this face, will be a tetraedron similar to the whole.*

232. Let the pentagonal pyramid S-FGHIK (*fig.*129) Fig.129.
be cut by a plane ABCDE, parallel to its base. Suppose SQ to be a perpendicular drawn from the summit to the base produced, the cutting plane will pass through the point P. The pentagon ABCDE will be the base of the smaller pyramid whose height is SP, as SQ is the height of the larger. The sides AB, BC, CD, DE, and EA,

Fig. 129. are respectively parallel to the sides FG, GH, HI, IK, and KF, of the base of the larger pyramid; and being parallel and directed the same way, the angles made by these sides are respectively equal to the angles of the base FGHIK. Now because each triangular face of the larger pyramid is cut parallel to its base, we have the proportions $\frac{SF}{SA} = \frac{FG}{AB} = \frac{SG}{SB}$, $\frac{SG}{SB} = \frac{GH}{BC} = \frac{SH}{SC}$, &c.; from which we derive the following; $\frac{SF}{SA} = \frac{SG}{SB} = \frac{SH}{SC}$, &c.; and $\frac{FG}{AB} = \frac{GH}{BC}$, &c., that is, the triangular faces of the two pyramids, have their sides proportional, and are therefore similar triangles; and the bases of the two pyramids have their sides proportional and their angles equal, they are consequently similar pentagons. These two pyramids are contained by similar planes similarly arranged, and the inclination of any two planes in the less pyramid, is equal to the inclination of the corresponding planes in the larger. The diedral angles made by similar triangular faces, are equal; and the corresponding diedral angles at the bases are equal, because the corresponding triangular faces are in the same plane, and the bases parallel. From the above analysis we derive the following truths:

(1.) *If any pyramid be cut by a plane parallel to its base, the smaller pyramid thus separated, is similar to the whole.*

(2.) *Any section of a pyramid parallel to its base, is similar to the base. All parallel sections in a pyramid, are similar polygons.*

233. We can find the height of a pyramid when we know the dimensions of a *trunk* or *frustum* of the pyramid, as FGHIK-ABCDE, which remains after cutting off the superior part S-ABCDE, and whose bases are parallel to each other; for taking the proportion $\frac{FG}{AB} = \frac{SQ}{SP}$, we have (81) $\frac{FG - AB}{AB} = \frac{SQ - SP}{SP} = \frac{PQ}{SP}$, which gives, $SP = \frac{AB \times PQ}{FG - AB}$; and adding SP to PQ, we have the height, and therefore all the dimensions of the entire pyramid.

234. We had $\frac{SF}{SA} = \frac{FG}{AB}$; but FG and AB are homologous sides of the similar sections FGHIK, ABCDE, of the pyramid. These sections being similar, are to each other as the squares of their homologous sides, that is, $\frac{FGHIK}{ABCDE} = \frac{(FG)^2}{(AB)^2}$; but $\frac{(FG)^2}{(AB)^2} = \frac{(SF)^2}{(SA)^2}$; and because QF and PA are parallel, we have $\frac{SF}{SA} = \frac{SQ}{SP}$; and $\frac{(SF)^2}{(SA)^2} = \frac{(SQ)^2}{(SP)^2}$; consequently we have $\frac{FGHIK}{ABCDE} = \frac{(SQ)^2}{(SP)^2}$. We say, therefore, that—*In any pyramid, sections made by parallel planes, are to each other as the squares of the distances of these planes from the summit.* Fig.129.

235. We had $\frac{SF}{SA} = \frac{SG}{SB} = \frac{SH}{SC}$, &c. We have also $\frac{SF}{SA} = \frac{SQ}{SP}$; we say, therefore,—*Similar pyramids have their homologous edges proportional; and these proportional to their heights.* The same may be proved of all other homologous measures; whence we have the more general truth—*In similar pyramids homologous dimensions are proportional.*

236. *Remark.* As the homologous faces of similar pyramids, are similar polygons; it follows that—*In similar pyramids the homologous faces are to each other, as the squares of the homologous edges, as the squares of their heights, or as the squares of their homologous dimensions generally.*

237. By drawing the diagonals FH, FI, in the base of the pyramid, and passing planes through these diagonals and the summit, we cut the pentagonal pyramid into three tetraedrons. And a slight inspection of any polyedral body whatever, will be sufficient to show, that it may be divided into tetraedrons, in a similar manner. And consequently—*If we have two similar polyedrons, by passing planes through them both in the same manner, we may divide them into the same number of similar tetraedrons.* The tetraedrons must be similar; for the bodies being similar, and the cutting planes passing through corresponding parts, these planes will have the same inclinations to the other faces and to each other, in the one

case as in the other. We say also—*When two polyedrons are composed of the same number of similar tetraedrons, similarly disposed, they are similar.*

From the preceding article, we infer that—*In similar polyedral bodies, homologous dimensions are proportional; and homologous faces and corresponding sections, are to each other as the squares of homologous dimensions in the bodies.*

238. Let $A$, $B$, $C$, $D$, $E$, $F$, be the faces of any polyedron; and $a$, $b$, $c$, $d$, $e$, $f$, be the corresponding faces of a similar polyedron; let $M$ be any diagonal in the first, and $m$ the corresponding diagonal in the second: We have $\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{D}{d} = \frac{E}{e} = \frac{M^2}{m^2}$. But in a series of equal fractions, the sum of the dividends contains the sum of the divisors as many times as one dividend contains its divisor; that is $\frac{A+B+C+D+E+F}{a+b+c+d+e+f} = \frac{M^2}{m^2}$. But the sum of all the faces of a polyedron, constitutes its *area*, or whole amount of surface; therefore—*The areas of similar polyedrons are to each other as the squares of their homologous dimensions.*

239. Among polyedral bodies, that which demands a particular examination is the *prism.* *A* prism *is a body which has two opposite faces equal and parallel, and all*
Fig. 130. *its other faces parallelograms.* In figure 130, the body ABCDEFGHIK, (frequently designated by AG) is a prism; the pentagon ABCDE is its *base.* The perpendicular distance of the base from its opposite face, is called the *height* of the prism. The body is formed by drawing from each of the angles of the base, the parallel lines AF, BG, CH, &c. above the plane of this base and terminated by the plane FGHIK, parallel to ABCDE. The lines AF, BG, &c. taken two and two, determine the *lateral* faces of the prism; and being parallel lines comprehended between parallel planes, they determine these faces to be parallelograms. The pentagon FGHIK, which forms what is sometimes called the *superior* base of the prism, is contained by lines respectively equal to those of the *inferior* base ABCDE (60); they are also parallel to them respectively; consequently the angles are equal, and the two polygons, having the homolo-

gous sides equal and their homologous angles equal, are equal to each other. It is evident, from similar considerations, that—*Any plane section* LMNOP *of the prism, parallel to its base, will be equal to its base; consequently, plane sections of a prism, parallel to each other, are equal.*

240. In any prism, each of the polyedral angles is composed of three plane angles. A prism in which the *lateral* faces are rectangles, is called *a right* prism; all others are called *oblique* prisms.

241. Suppose, in the prism AI (*fig.* 130), we have the base given, and the two faces, AG and BH; if we apply the two faces to the base, in the same manner as they are disposed in the figure, to form the triedral angle whose vertex is at B, the magnitude of these angles will determine the inclinations of the planes. These determine the positions of the points F, G and H; the three points H, C, D, determine the plane CI; and this being a parallelogram fixes the point I. In a similar manner it may be shown that all the other parts of the prism are fixed by the above conditions. We say, therefore—*All the parts of any prism are determined by the three planes forming either of the triedral angles, and the disposition of those planes;* therefore—*If two prisms have the three planes forming a triedral angle in one, equal respectively to the three planes which form a triedral angle in the other, and disposed in the same order, the prisms are equal. If the planes are respectively equal, but the arrangement different, the prisms will be symmetrical.* Of *right* prisms, those which are *symmetrical* are also *equal.* Fig.130.

242. The prism ABCDEFGH (usually designated by AG) (*fig.* 131), whose bases are parallelograms, is called *a parallelopiped;* it has its opposite faces equal and parallel. Its base ABCD is a parallelogram, therefore AD is equal and parallel to BC; ABFE is a parallelogram, consequently AE is equal and parallel to BF; but when two lines which cut each other are parallel to two other lines which cut each other, the plane determined by the first two is parallel to the plane determined by the second two (211); therefore the plane AEHD, is parallel to BFGC; and they are evidently equal, their sides and angles being equal. In the same manner we may show that the two other opposite faces are equal and parallel, Fig.131.

We say, therefore—*In a parallelopiped any two opposite faces are equal and parallel;* and—*A prism contained by six planes, of which each is parallel to one of the others, is a parallelopiped.*

243. By comparing the plane angles at two opposite vertices in any parallelopiped, the learner will perceive that they are equal respectively, being the corresponding angles in equal parallelograms; and as each is a triedral angle, the inclination of two faces in the one, must be equal to that of the corresponding faces in the other; but the plane angles in one are in the inverse order of those of the other; it will therefore follow, that—*In every inclined parallelopiped the opposite triedral angles are symmetrical.*

244. A parallelopiped is called *rectangular* when each of its faces is a rectangle. This will be the case, if, at one of the triedral angles, each of the edges is perpendicular to the plane of the other two. The rectangular parallelopiped of which each of the faces is a square, is called a *cube*. It is also called a *regular hexaedron*, being contained by six equal regular polygons, and having its diedral angles all equal, and its triedral angles equal.

245. The bases of a parallelopiped are parallelograms; consequently the sum of the squares of their four diagonals, is equal to the sum of all the squares described upon the eight edges at the bases (184). If through opposite diagonals in the bases, and therefore through opposite
Fig.131. vertices of the body (*fig.* 131), two planes are passed, the two *sections* thus made will be two parallelograms, four sides of which are the four diagonals of the bases, and their other sides the four lateral edges of the body. In these two sections there are four diagonals, which are evidently the *diagonals of the parallelopiped;* and *the sum of the squares of these diagonals, must be equal to the sum* of the squares of the sides (184), that is, to the sum *of the squares of the twelve edges of the parallelopiped.* Therefore—*In a rectangular parallelopiped, the square of one of the diagonals is equal to the sum of the squares of the three edges which meet at the same vertex.* And—*In the cube, the square of a diagonal is equal to* three times *the square of one of the edges.*

246. The cube is distinguished among bodies, as the square among surfaces. All the plane angles in its faces are right-angles; all its diedral angles are right angles;

each of its triedral angles is composed of three right-angles; and all its edges are equal. This character of its angles, and the equality of its sides, give to its form a remarkable simplicity; on which account it has been selected as a *unit* in the measure of bodies.

### *Of the Volume of Bodies, and its Measure.*

247. By the *volume* of a body, we mean *the quantity of space* which the body occupies, or the whole amount of space included by its surface. The *volume* of a body is very distinct from its *form*. Two bodies may be equal in volume and very different in form. A cubic cistern may be constructed which shall contain the same quantity of water as a hogshead. If the water be drawn from the hogshead into the cistern, the quantity of water remaining the same, the whole amount of space which the water occupies will be the same; the volume is not changed, though the form is very different. We see, therefore, that two very different geometrical bodies may be equal in volume. If two bodies are equal in volume, they are called *equivalent bodies*.

248. Suppose we have two parallelopipeds, AG and AL, standing upon the same base ABCD, and having their superior bases in the same plane parallel to ABC Fig.132. (*fig.* 132). We observe first, that the whole figure consists of the parallelopiped AG, and the triangular prism BFK-CGL, and also of the parallelopiped AL, and the triangular prism AEI-DHM. If, therefore, we can show that these two triangular prisms are equal, as one of them added to one of the parallelopipeds, makes the same body as the other prism added to the other parallelopiped, the two parallelopipeds must be equal in volume.

The triangles AEI, BFK, have their sides AE and BF equal, being opposite sides of the same parallelogram; for the same reason AI = BK; as these equal lines are also parallel the angles EAI and FBK are equal; the two triangles are therefore equal by the second case; the faces EC and FD are also equal being opposite faces of the same parallelopiped (242); EF is equal to IK, being each equal to AB; if from each of these we take IF, we have EI = FK; EH is also equal and parallel to FG, therefore the angle IEH is equal to the angle KFG;

and the two parallelograms EM and FL, having two adjacent sides and the contained angle the same, are equal; we therefore have the three faces containing the triedral angle whose vertex is at E, respectively equal to the three faces which contain the corresponding triedral angle in the other prism; the two prisms are therefore equal, and the parallelopipeds AG and AL are equivalent.

In a similar manner we can show that the perallelopiped ABCD-IKLM, is equivalent to ABCD-EFGH
Fig.133. (*fig.* 133), they being upon the same base ABCD, and having their superior bases in the same plane. So also it may be shown that the parallelopiped ABCD-NOPQ, whose edges are perpendicular to the base is equivalent to the parallelopiped ABCD-EFGH; therefore the parallelopiped ABCD-IKLM is equivâlent to ABCD-NOPQ.

But this parallelopiped AP, may be changed into the equivalent parallelopiped ABRS-NOTU, whose bases
Fig.134. are rectangles (*fig.* 134) as the triangular prism BCR-OPT is equal to the prism ADS-NQU.

It is evident that all the faces of this last parallelopied, are rectangles; and each of these inclined parallelopipeds is equivalent to this *rectangular parallelopiped.* We therefore say—*Any parallelopiped may be changed into an equivalent rectangular parallelopiped, having the same height and an equivalent base.*

249. Any parallelopiped may be divided into two equivalent triangular prisms by a plane passing through two opposite lateral edges; for each base of the triangular prism is half the base of the parallelopiped, and their heights are the same. If, therefore, an *inclined* parallelopiped may be changed into an equivalent rectangular parallelopiped of the same base and height; the triangular prisms which compose the inclined parallelopiped, must be respectively equivalent to the triangular prisms which compose the right parallelopiped, as in either case each prism is one half of the parallelopiped. Whence it appears that—*Any inclined triangular prism may be changed into an equivalent right prism of the same height and an equivalent base.*

250. Any prism of more than three lateral faces, may be cut into triangular prisms, by passing planes through opposite lateral edges. These prisms will have the same volume when their bases are the same and their heigh

the same, whether their edges are inclined to the base or not (249). We say therefore—*Any two prisms having equivalent bases and the same height are equivalent, or equal in volume.*

251. In a comparison of pyramids, we will take the triangular pyramid, or tetraedron.

Let us suppose that the two pyramids S-ABC, *s-a b c*, (*fig.* 135) have the same height and equivalent bases in the same plane; let their height be represented by AH, perpendicular to the plane of their bases. Divide AH into any number of *equal* parts, at the points *w*, *x*, *y*, *z*, and through these points, suppose planes to be passed parallel to the plane of the bases; the corresponding sections of the two pyramids, made by these planes, will beequivalent (234), namely DEF to *d e f*, GHI to *g h i*, &c. Fig.135.

Upon the upper side of the triangles ABC, DEF, GHI, &c. as bases, construct prisms, which will be *exterior* to the corresponding segment of the pyramid; and upon the lower side of *d e f*, *g h i*, *k l m*, &c., construct prisms which will be *interior* to the corresponding segments. It is manifest that the sum of all the exterior prisms of the pyramid S-ABC, is greater than the pyramid; and that the sum of all the interior prisms in the pyramid *s-a b c* is less than the pyramid; therefore the pyramid S-ABC minus the pyramid *s-a b c*, must be *less* than the difference between the sum of the interior and the sum of the exterior prisms. But the prisms all have the same height; and therefore, those upon equivalent bases are equivalent; consequently the sum of all the interior prisms in the pyramid *s-a b c*, is equal to the sum of all the exterior prisms about the other pyramid, excepting the lower prism whose base is ABC. This prism, therefore, is the difference between the sum of the exterior prisms about the pyramid S-ABC, and the sum of the interior prisms in the pyramid *s-a b c*.

If we suppose the pyramids to be cut by planes only half the distance asunder before supposed, and prisms constructed as above; the prism whose base is ABC will still be the difference between the sums of the two series of prisms; but having the same base and a less height, its volume must be less. It is evident that as we increase the number of prisms in the series, the volume of that prism which is equal to the difference of the two series, must be continually diminished; whence we may

continue the process till this difference is as small as we please; but this difference will always be greater than the difference of the two pyramids, which must consequently be infinitely small, that is, *nothing*. We say, therefore—*Two triangular pyramids which have the same height and equivalent bases, are equal in volume.*

252. As every pyramid of more than three lateral faces, may be cut into triangular pyramids of the same height, by planes passing through the summit and the diagonals of the base, this law of triangular pyramids will be true of all pyramids composed of triangular pyramids. We say, therefore, that—*Pyramids of the same height and equivalent bases, are equal in volume.*

253. We have seen that polyedral bodies may be cut into tetraedrons (or triangular pyramids) by planes passing through them diagonally (237). Suppose we have
Fig.136. the triangular prism ABC-DEF (*fig.* 136). First, pass a plane through the vertex E and the edge AC of the base; it will cut off the tetraedron E-ABC, whose base and altitude are the same as those of the prism. Suppose the remainder ACDEF, to be cut through the vertices C, D, E, it will be divided into two tetraedrons, one of which, CDEF, may be considered as having C for its summit, and DEF, one of the bases of the prism, for its base; this tetraedron has also its base and height equal to those of the prism. The remaining segment EACD, may be considered as a tetraedron whose summit is at E, and whose base is ACD; but as the edge BE is parallel to the opposite face of the prism, the point B is at the same distance from the plane of the opposite face, as the point E is; we may therefore substitute for the tetraedron EACD, the equivalent tetraedron BACD, having the same base ACD, and the same height. But in the tetraedron BACD, we may take ABC for the base, and the point D for the summit; this tetraedron has also the same base and the same height with the prism. We have, therefore, three equivalent tetraedrons composing this prism. We, therefore say—*A tetraedron is equivalent to one third of a triangular prism of the same base and height.*

254. By reviewing the analysis of the last article, it will be perceived that the body ABC-DEF is equivalent to three tetraedrons which have ABC for a common base, and their vertices respectively in the three points D, E,

and F. If, therefore, the body were a *truncated* prism, that is, if the plane DEF were *inclined* to the base, we should say—*A truncated triangular prism is equivalent to three pyramids whose common base is the base of the prism, and whose heights are respectively equal to the distances of the opposite vertices from the base.*

255. As all pyramids of more than three lateral faces may be cut into tetraedrons of the same height, the sum of whose bases will be the base of the pyramid; and as the prisms constructed upon the bases of these tetraedrons taken together, will constitute a prism of the same height, constructed upon the base of the pyramid in question; and as each of the tetraedrons is one third of the corresponding triangular prism, the sum of all the tetraedrons will be one third of the sum of all the triangular prisms. We therefore say (249)—*Any pyramid is one third of a prism of the same base and height, or one third of a rectangular parallelopiped of the same height and an equivalent base.*

256. Let us now compare together bodies of different heights. Suppose the two rectangular parallelopipeds AG and AL (*fig.* 137) to have the same base ABCD; Fig.137. and AE and AI, for their heights. Supppose these heights to be to each other in any definite ratio, for instance in the ratio of 9 to 5. Divide AE into 9 equal parts, AI will contain 5 of them; through these points of division pass planes parallel to the base; they will divide the parallelopiped AG into 9 rectangular parallelopipeds of equal bases and equal heights; and as the parallelopiped AL contains 5 of these, the two parallelopipeds will be to each other as 9 to 5; that is, as AE to AI. We should have the same result if their heights were in any other commensurable ratio; the bodies would be in the ratio of their heights.

Suppose their heights to be incommensurable; we may divide one of the lines into equal parts so small that the difference between one of these parts and the remainder of the division of the other line by such part, is less than any given difference; we thus find an approximate ratio of the two bodies, which is indefinitely near to the ratio of the two lines; but which ratio we never pass, on which side soever we commence the approximation; we therefore infer that the ratio of the heights is, in this case also, the ratio of the two bodies. We therefore say—*Two*

*rectangular parallelopipeds of the same base, are to each other in the ratio of the heights.*

257. Let us now compare the two rectangular parallelopipeds AG and IP (*fig.* 138) of different bases and different heights. Upon the edge IN take II′ = AE, and through this point suppose a plane section I′L′ parallel to the base; and upon BC take BC′ = IM, and through the point C′ suppose the plane section C′H′ parallel to the face AF. We shall then have the parallelopipeds AG′ and IL′ whose bases are the equal rectangles AH′ and IM′, and whose heights are the edges AB and IK; we shall therefore have (256) $\frac{vol.\ AG'}{vol.\ IL'} = \frac{AB}{IK}$; and by comparing the parallelopipeds AG and AG′, considered as having for their common base the rectangle AF, we obtain $\frac{vol.\ AG}{vol.\ AG'} = \frac{AD}{AD'}$. By multiplying these two proportions in order, omitting the factor AG′ common to the numerator and denominator, and substituting for AD′ its equal IM, we make a comparison of the paralleloped AG and IL′; this gives $\frac{vol.\ AG}{vol.\ IL'} = \frac{AB \times AD}{IK \times IM}$. Comparing now the parallelopipeds IL′ and IP, having the same base IKLM, we obtain the proportion $\frac{vol.\ IL'}{vol.\ IP} = \frac{II'}{IN}$. Multiplying together these two equations, omitting the common factor IL′ in the two terms of the first ratio, and substituting in the second AE for its equal II′, we obtain the following: $\frac{vol.\ AG}{vol.\ IP} = \frac{AB \times AD \times AE}{IK \times IM \times IN}$. We therefore say—*Any two rectangular parallelopipeds are to each other as the products of the three edges which meet at one of the vertices;* that is, *as the products of their three dimensions.*

258. To illustrate this result, we take as the unit of the measure of volume, the cube whose side or edge is a unit in the measure of lines. Let $a\,g$ be this cube (*fig.* 139). If we compare this small cube whose side is 1, with the parallelopiped AG, we shall have

$$\frac{vol.\ AG}{vol.\ a\,b} = \frac{AB \times AD \times AE}{1};$$ that is, the par

allelopiped AG contains the cube whose side is $a\,b$ as many times as the product of the lines AB, AD, AE, referred to the common measure $a\,b$, contains *unity*. And this is what is meant, when it is said that—*The measure of volume in a rectangular parallelopiped, is the product of its three contiguous edges.* Fig.189.

**259.** We observe that the product AB $\times$ AD expresses the number of squares, whose side is unity [$a\,b$], contained in the base AC; that is, expresses the area of the base; we say therefore, that—*The volume of a rectangular parallelopiped has for its measure the product of its base by its height.*

**260.** As any parallelopiped may be changed into a rectangular parallelopiped of the same height and an equivalent base (248), it follows that—*Any parallelopiped has for the measure of its volume, the product of its base by its height.*

**261.** As a rectangular parallelopiped may be divided into two equivalent triangular prisms of the same height with the parallelopiped, whose bases are each half the base of the parallelopiped (249); it follows, that—*The volume of a triangular prism has for its measure the product of its base by its height.*

**262.** As every prism of more than three lateral faces may be considered as composed of triangular prisms of the same height with itself, the sum of whose bases make the base of the prism (250), we say—*The volume of every prism has for its measure the product of its base by its height.* And consequently (255)—*The volume of any pyramid has for its measure, the product of one third of the base by the height.*

*Remark.* To obtain the volume of any frustum of a pyramid, find the volume of the entire pyramid, and subtract from it the volume of the partial pyramid cut off.

**263.** Let $P$ represent any prism, $B$ its base and $H$ its height; we shall have the formula $P = B \times H$; let $P'$ denote a similar prism, $B'$ its base, and $H'$ its height; this gives $P' = B' \times H'$. If we compare these, we have $\frac{P}{P'} = \frac{B \times H}{B' \times H'}$. But in similar polyedrons, homologous faces are proportional to the squares of their homologous measures (237); the bases are then as the

squares of their heights; this gives $\frac{B}{B'} = \frac{H^2}{H'^2}$; we may therefore substitute for $\frac{B}{B'}$ in the proportion above, the equal ratio $\frac{H^2}{H'^2}$; this will give us $\frac{P}{P'} = \frac{H^2 \times H}{H'^2 \times H'} = \frac{H^3}{H'^3}$; that is—*Similar prisms are to each other as the third powers of their heights,* or *as the cubes of their homologous measures generally.*

264. Let $p$ represent a pyramid, $b$ its base and $h$ its height; $p'$ another similar pyramid, $b'$ its base, and $h'$ its height; for the first of these pyramids, we have the expression $p = \frac{1}{3} b \times h$; and for the second, $p' = \frac{1}{3} b' \times h'$. By comparing these, we obtain the proportion $\frac{p}{p'} = \frac{\frac{1}{3} b \times h}{\frac{1}{3} b' \times h'}$; and by a process of reasoning similar to the above, we obtain $\frac{p}{p'} = \frac{h^3}{h'^3}$; that is—*Similar pyramids are to each other as the cubes of their heights;* or, *as the third powers of their homologous dimensions generally.*

265. As all polyedral bodies may be considered as composed of pyramids; we have from the above rule, a method of ascertaining the measure of their volume. And as similar polyedral bodies are composed of the same number of similar tetraedrons or triangular pyramids, we infer that—*Similar polyedrons are to each other as the cubes described upon their homologous measures.*

# PART SECOND.

## Section II.—*Of the Round Bodies.*

266. Round bodies are those which are produced by the revolution of any plane figure about a straight line; they are called *bodies of revolution.* Those usually discussed in the elementary treatises, are the *right cone*, the *right cylinder* and the *sphere.*

The right cone is *generated* by the revolution of a right-angled triangle about one of the sides containing the right-angle, as the triangle SCA (*fig.* 140) about the side SC. The hypothenuse SA by this motion generates the *conical surface.* Each point in the hypothenuse, describes the circumference of a circle whose centre is in the line SC; the circle generated by the line CA is called the *base* of the cone. The line CS, upon which the generating triangle turns, is called the *axis* of the cone. The point S is called the *summit* or *apex* of the cone. Fig.140.

It is evident that a plane passing through the axis will cut the conical surface in two straight lines. A plane perpendicular to the axis, will have the circumference of a circle for its section of the conical surface.

267. The cone just described is the right cone whose base is a circle. The *inclined* cone with a circular base (*fig.* 141) may be considered as generated by the motion of a straight line, as AS, one point of which, as S, being fixed, the other part being carried round the circumference of a circle, as ADB, situated in a plane which does not pass through S. The straight line CS joining the apex with the centre of the base, is here also called the *axis.* This cone is also called an *oblique* or *scalene* cone. Fig.141.

268. In the right cone (*fig.* 140) the similar triangles ACS, A′C′S, which give the proportion $\frac{AC}{A'C'} = \frac{CS}{C'S} = \frac{AS}{A'S}$, show that the radii of the circles ADB, A′D′B′, are proportional to their distances from the apex of the cone; Fig.140.

Fig.140. but the circumferences of circles being as their radii (148), and their areas being as the squares of their radii 192), we have $\frac{circ.\ \text{ADB}}{circ.\ \text{A}'\text{D}'\text{B}'} = \frac{\text{AC}}{\text{A}'\text{C}'} = \frac{\text{CS}}{\text{C}'\text{S}} = \frac{\text{AS}}{\text{A}'\text{S}}$, and $\frac{area\ \text{ADB}}{area\ \text{A}'\text{D}'\text{B}'} = \frac{(\text{AC})^2}{(\text{A}'\text{C}')^2} = \frac{(\text{CS})^2}{(\text{C}'\text{S})^2} = \frac{(\text{AS})^2}{(\text{A}'\text{S})^2}$.

In figure 141, supposing A′D′B′ to be parallel to the base, we have also $\frac{circ.\ \text{ADB}}{circ.\ \text{A}'\text{D}'\text{B}'} = \frac{\text{AC}}{\text{A}'\text{C}'} = \frac{\text{CS}}{\text{C}'\text{S}} = \frac{\text{AS}}{\text{A}'\text{S}} = \frac{\text{OS}}{\text{O}'\text{S}}$, and $\frac{area\ \text{ADB}}{area\ \text{A}'\text{D}'\text{B}'} = \frac{(\text{AC})^2}{(\text{A}'\text{C}')^2} = \frac{(\text{CS})^2}{(\text{C}'\text{S})^2} = \frac{(\text{AS})^2}{(\text{A}'\text{S})^2} = \frac{(\text{OS})^2}{(\text{O}'\text{S})^2}$. We see, therefore, that it is with cones as with pyramids (232).

(1). *If any cone be cut by a plane parallel to its base, the smaller cone cut off is similar to the whole.*

(2). *Similar cones have their homologous dimensions proportional, and their bases proportional to the squares of their homologous lines; and*

(3). *In any cone, parallel sections are to each other as the squares of their distances from the apex.*

269. *Remark.* When we have the dimensions of any *truncated* cone whose bases are parallel, as ADB, A′D′B′, we can find by a process analogous to that of article 233, the dimensions of the entire cone, and also of the superior cone, S - A′D′B′.

Fig.142. 270. If we inscribe in the base of the cone (*fig.* 142) a regular polygon, and circumscribe also about the base a similar polygon; by drawing straight lines from the apex of the cone to the vertices of these polygons, we shall have a pyramid *inscribed* in the cone, and also a pyramid *circumscribed* about the cone. The figure, not to make confusion, exhibits but one of the lateral faces of each pyramid. The circumscribed pyramid exceeds the inscribed pyramid by a certain magnitude; but by increasing the number of sides in the polygons which are their bases, we increase the number of lateral faces in the pyramids. The inscribed pyramid is increased, and the circumscribed pyramid is diminished by this process, which may be continued till the difference between the two pyramids is less than any assignable magnitude. By this process, also, the lateral surfaces of the two pyramids are made to approach indefinitely near to

each other. The lateral surface of the inscribed pyramid can never be greater than the conical surface, and the lateral or *convex* surface of the circumscribed pyramid can never be less than the conical surface. The difference between the convex surface of the circumscribed pyramid and the conical surface, must always be less than the difference between the two pyramidal surfaces which we suppose less than any assignable magnitude. Fig.142.

271. The convex surface of the pyramid circumscribed about a right cone, is composed of a certain number of triangles, which, on account of the regularity of the base, and the summit being in a straight line perpendicular to the middle of the base, are isosceles and equal. The common height of all these triangles is SG the side of the cone; one half the product of SG multiplied by the sum of their bases, will give their area. The sum of the bases of these triangles is the perimeter of the polygon; therefore, if we denote this perimeter by $P$, we shall have for the *area* of the pyramid $\frac{1}{2}P \times$ SG. This perimeter exceeds the circumference of the circle, and therefore this product exceeds the product $\frac{1}{2}$*circ.* $\times$ SG by an indefinitely small magnitude which we designate by $d$; and denoting the area of the cone by $A$, and the excess of the pyramidal surface over the conical surface by $m$; we shall have the equation $A + m = \frac{1}{2}$*circ.* $\times$ (SG) $+ d$. The area of the inscribed pyramid, calling $p$ the perimeter of its base, will be $\frac{1}{2}p \times sg$; and denoting the excess of the conical surface over the lateral surface of this pyramid by $m'$; and also denoting by $d$ the excess of the product $\frac{1}{2}$*circ.* $\times$ (SG) over the product $\frac{1}{2}p \times (sg)$, which may be less than any assignable quantity, as by increasing the sides of the polygon, the perimeter becomes sensibly confounded with the circumference, and $sg$ with SG, the side of the cone: we shall have $A - m' = \frac{1}{2}$*circ.* $\times$ (SG) $- d$; we see, therefore, that if any magnitude, however small, be added to the area of the cone, something must be added to the product $\frac{1}{2}$*circ.* $\times$ (SG) to make the equation; and if any magnitude, however small, be subtracted from this area, a corresponding quantity must be subtracted from this product to balance the equation; from which it follows that $\frac{1}{2}$*circ.* $\times$ (SG) expresses neither more nor less than the area of the conical surface. Wherefore—*The con-*

Fig.142. *vex surface of a right cone, is measured by half the product of the circumference of the base multiplied by the side.*

272. To find the area a *truncated cone.* Subtract from the area of the entire cone the area of the smaller cone cut off. Calling $A'$ the area of the frustum
Fig.143. ADB-A'D'B' (*fig.* 143), $A$ the area of the entire cone, and $a$ the area of the less, we shall have $A' = A - a = \frac{1}{2}(circ.\ \mathrm{OA}) \times (\mathrm{SB}) - \frac{1}{2}(circ.\ \mathrm{O'A'}) \times (\mathrm{SB'})$, or $A' = \frac{1}{2}(circ.\mathrm{OA}) \times (\mathrm{SB}) - \frac{1}{2}(circ.\mathrm{OA}) \times (\mathrm{SB'}) + \frac{1}{2}(circ.\ \mathrm{O'A'}) \times (\mathrm{SB}) - \frac{1}{2}(circ.\ \mathrm{O'A'}) \times (\mathrm{SB'}) = \frac{1}{2}(circ.\ \mathrm{OA}) \times (\mathrm{SB} - \mathrm{SB'}) + \frac{1}{2}(circ.\mathrm{O'A'}) \times (\mathrm{SB}-\mathrm{SB'}) = \frac{1}{2}(circ.\ \mathrm{OA} + circ.\mathrm{O'A'}) \times (\mathrm{BB'})$. Hence—*The area of the frustum of a right cone has for its measure the product of its side multiplied by half the sum of the circumferences of its bases.* Or supposing a section A″D″B″E″, at equal distances from the two bases of the frustum; the similar triangles SOA, SO″A″, SO′A′, give $\frac{\mathrm{SO}}{\mathrm{OA}} = \frac{\mathrm{SO''}}{\mathrm{O''A''}} = \frac{\mathrm{SO'}}{\mathrm{O'A'}}$; whence (82) $\frac{\mathrm{SO}-\mathrm{SO''}}{\mathrm{OA}-\mathrm{O''A''}} = \frac{\mathrm{SO''}-\mathrm{SO'}}{\mathrm{O''A''}-\mathrm{O'A'}}$, that is, $\frac{\mathrm{O''O}}{\mathrm{OA}-\mathrm{O''A''}} = \frac{\mathrm{O'O''}}{\mathrm{O''A''}-\mathrm{O'A'}}$. These numerators are equal by construction; the denominators are therefore equal; and OA is as much greater than O″A″, as O″A″ is greater than O′A′; but these radii are as their circumferences. The circumference A″D″B″E″ is therefore an arithmetical mean between the other two; and is consequently equal to half their sum. Hence—*The area of the frustum of a cone, has for its measure the product of its side multiplied by the circumference of the plane section made at equal distances from its bases.*

*Remark.* By substituting the apex of the cone for the superior base, the first formula gives the area of the entire cone.

273. We have seen (270) that two pyramids may be constructed the one inscribed in a cone and the other circumscribed about it, such that the difference of their volumes shall be less than any assignable magnitude. The cone, therefore, may be considered as a pyramid of an infinite number of lateral faces; it will therefore have for the measure of its volume, one third of the product of

its height by its base. We shall however give this a different proof.

The common height of these pyramids will be SO the height of the cone (*fig.* 142). If we call $B$ the base of the cone, and $d$ the difference between the base of the cone and the base of the circumscribed pyramid, and $d'$ the difference between the base of the cone and the base of the inscribed pyramid, $V$ the volume of the cone, $m$ the excess of the volume of the circumscribed pyramid over the volume of the cone, and $m'$ the excess of the volume of the cone over the volume of the inscribed pyramid. This will give $V + m = \frac{1}{3}(\text{SO}) \times (B + d)$; $V - m' = \frac{1}{3}(\text{SO}) \times (B - d')$. That is, if any thing be added to the volume of the cone, something must be added to the product $\frac{1}{3}(\text{SO}) \times B$, to balance the equation; and if any magnitude however small be subtracted from the volume of the cone, a corresponding quantity must be subtracted from this product, to preserve the equation; from which it follows that $\frac{1}{3}(\text{SO}) \times B$, expresses neither more nor less than the volume of the cone; and we say—*The volume of a cone has for its measure, one third of the product of the base multiplied by the height.* Fig.142.

174. If the radius of the base of a cone be designated by $R$, and the height of the cone by $H$; we shall have for the base of the cone $\pi . R^2$. Denoting its volume by $V$, we have $V = \frac{1}{3}\pi . R^2 \times H$.

*Remark.* To find the volume of a truncated cone, subtract the volume of the less from the volume of the entire cone.

275. If we conceive the rectangle AC C′A′ (*fig.* 144) to revolve about one of its sides CC′, it will generate the body ADB-A′D′B′, called a *right cylinder.* The straight line AA′ will, by this revolution, generate the *cylindrical surface.* Fig.144.

Any point, as A″, in this straight line, will describe the circumference of a circle, as A″D″B″, equal and parallel to the circle ADB called the *base* of the cylinder and generated by AC the base of the rectangle. For A″C″ perpendicular to CC′, and AC also perpendicular to CC′, will, by this motion, describe plane circles perpendicular to CC′ (199); they are therefore parallel; A″C″ being equal to AC, and they being the radii of these circles, the circles are equal. From which it follows that—*Any section of a right cylinder parallel to its*

Fig.144. *base, is a circle equal to its base. Any plane section through the axis or parallel to the axis, is a rectangle.*

The straight line A′C′, also perpendicular to CC′, generates the circle A′D′B′, also equal and parallel to ADB; this we call the superior base. The line CC′ about which the rectangle revolves, is called the *axis* of the cylinder, and is perpendicular to the bases.

276. If we conceive a circle, as ADB, around
Fig.145. which a straight line, as AA′ (*fig.* 145) moves always parallel to itself, but oblique to the plane of the circle; this line will generate the surface of an *oblique* or *inclined* cylinder. Or we may conceive this cylinder to be generated by the motion of the circle ADB parallel to itself, along the straight line AA′, every point in the circle describing a straight line by this motion. The circumference of the circle will generate the cylindrical surface; the centre of the circle will generate the axis CC′; and as the generating circle is always parallel to itself, every section of this cylinder parallel to the base, will be a circle equal to the base. Any section of this cylinder through the axis or parallel to the axis, is a parallelogram. Sections parallel to the axis and having the same inclination to the base as the axis has, are rectangles.

277. We have here given two methods of generating the inclined cylinder. In the first, we call the straight line AA′ which generated the cylindrical surface, the *generatrix,* and the circle by whose circumference its motion was guided, we call the *directrix.* In the latter method we call the circle by which the cylinder was generated, the *generatrix;* and the straight line by which its motion was directed, the *directrix.* If in either of these two cases the straight line had been perpendicular to the plane of the circle, the body generated would have been a *right cylinder.**

278. A slight inspection will satisfy us that the cyl-

* These methods of defining a cylinder are preferable to the description given in Art. 274; for they will apply to all cylinders, if, instead of the term *circle,* we substitute *a plane surface embraced by a curve line*; as it is not an essential property of a cylinder, that any section of it should be a circle. Nor is it an essential property of a cone, that any section of it should be a circle.

inder belongs to the family of prisms; and therefore to find the area and volume of a cylinder we should proceed as we would with the prism in the same cases. That is, the area of any right cylinder has for its measure the height of the cylinder multiplied by the periphery of the base; and the volume of *any* cylinder is measured by the the product of its base multiplied by its height. This may be readily demonstrated.

279. If we inscribe a regular polygon in the circular base of the right cylinder (*fig.* 146) and circumscribe about Fig. 146. the circle a similar polygon, and make these two polygons the bases of two prisms the one inscribed in the cylinder and the other circumscribed about it; as the number of sides of these polygons may be so great that the difference between the polygons shall be as little as we please, it is manifest that we may erect upon the polygons two prisms, the one inscribed in the cylinder and the other circumscribed about it, whose difference shall be less than any assignable magnitude.

The area of the cylinder is less than the area of the circumscribed prism by an indefinitely small magnitude $m$, and greater than that of the inscribed prism, by an indefinitely small quantity $m'$. The perimeter of the base of the circumscribed prism exceeds the circumference of the base of the cylinder, by a quantity less than any assignable magnitude, which excess we denotede by $d$. This circumference exceeds the perimeter of the base of the inscribed prism by an indefinitely small excess which we designate by $d'$. Denoting the area of the cylinder by $A$ and the height by $H$, we shall have for the area of the exterior prism $A + m = H \times (circ. + d)$; and for the area of the interior prism, $A - m' = H \times (circ. - d')$; that is, if any magnitude, however small, be added to the area of the cylinder, something must be added to the product of its height multiplied by the circumference of the base, to express this area; and if any magnitude however small be subtracted from the area of this cylindrical surface, a corresponding magnitude must be subtracted from the above mentioned product, to express this area; that is—*The product of the height of the right cylinder multiplied by the circumference of the base*, (being the measure of an area neither greater nor less) *is the measure of the area of the cylindrical surface.*

280. The learner is requested to give an analogous

proof that—*The volume of a cylinder has for its measure the product of the area of its base multiplied by its height.*

*Remark.* It is manifest that this last proposition applies as well to oblique as to right cylinders, there being nothing in the reasoning which supposes the axis of the cylinder perpendicular to the base.

Fig.147. 281. If the semicircle ACB (*fig.* 147) revolve about its diameter AB, it will generate a *sphere;* the semicircumference generates the *spherical surface.* Every point in this semi-circumference, as D, generates the circumference of a circle whose centre is in the diameter AB (here called the *axis)* and whose plane is perpendicular to this axis. The extremities A, B, of the axis, are, in reference to this revolution, called *poles.* They are particularly the poles of all the circles whose circumferences are generated by the several points in the arc ACB of the revolving semicircle.

As every point of the arc of the semicircle is at the same distance from its centre, and as this centre does not move during the process of revolution; it follows that every point of the spherical surface is equally distant from the point O, the centre of the generating semicircle and also the *centre of the sphere.* It is also evident that any radius of the semicircle, in any part of its revolution, is also radius of the sphere; and that these radii are all equal.

282. If we suppose the sphere to be cut by a plane passing through its centre, it is perfectly manifest that the section will be a circle, whose radius will be the radius of the sphere.

But suppose the plane to pass through the sphere on one side of the centre; and let the section be DGFH; from O the centre of the sphere, draw OE perpendicular to this section, and draw radii to the several points in the periphery of this section; they will be equal oblique lines, and will therefore meet the plane DGFH, at equal distances from the perpendicular OE (206). Every point in this periphery is therefore at the same distance from E; the section is consequently a circle whose centre is E. And as this reasoning applies to any plane section on either side of the centre, we infer that—*Every plane section of a sphere is a circle;* and also that—*Radius*

*perpendicular to any plane section of the sphere, passes through its centre.* Fig.147.

283. *Remark.* As DE is always one side of a right-angled triangle whose hypothenuse is radius of the sphere, DE must be less than radius; the plane section whose radius is radius of the sphere, is called a *great circle;* other plane sections are called *less circles.*

284. As two great circles are planes, their intersection must be a straight line; and as each passes through the centre of the sphere, the line of intersection must pass through the centre of each of them, that is—*Two great circles mutually bisect each other.*

285. Any portion of a spherical surface embraced by the arcs of three great circles, is called a *spherical triangle.* As two circumferences of great circles bisect each other, if each angle of a spherical triangle is salient, each of the arcs must be less than a semi-circumference.

286. Let CMI be a spherical triangle, and draw the radii OC, OM, OI; it is evident that they determine a triedral angle whose vertex is at O, and whose plane angles are measured by the arcs CI, CM, IM. And as the sum of any two of these plane angles is greater than the third (220) the sum of the arcs which measure them must be greater than the third arc (113); that is—*The sum of any two sides of a spherical triangle, is greater than the third.* Whence we infer that—*The shortest way from one point to another, on the surface of a sphere, is in the arc of a great circle passing through these points.* For if we suppose it shorter to pass through any point out of this arc, that point and the two points proposed may be made the vertices of a spherical triangle, one side of which will be the arc first proposed and less than the sum of the other two; the inference is therefore manifestly true.

287. Suppose the two semi-circumferences ACB and AIB which meet each other in A and B, to be cut by the arc IM of another great circle; we shall have $MI < MB + BI$; and consequently $AM + AI + IM < AMB + AIB$; that is—*The sum of the three sides of any spherical triangle, is less than the circumference of a great circle.*

*Remark.* We might have deduced this truth from the limit of the magnitude of the plane angles which form the triedral angle at the centre of the sphere (222); this sum being less than four right angles, the sum of the

Fig.147. arcs which measure these angles. must be less than an entire circumference (27).

288. As any straight line drawn through any point in a curve line (in the plane of the curve) and oblique to a right-line tangent at that point, must cut the curve line (111), the tangent may be considered as representing the direction of the curve at that point. It follows from this, that—*The angle which any two curves make with each other at their intersection is neither more nor less than the angle which their respective tangents at this point make with each other.*

The angle CIM of the spherical triangle will therefore be the same as the angle which the tangents to the two arcs CI, MI, at the point I, make with each other. These tangents are perpendicular to radius IO (111), and are in the planes of the two circles respectively; but being in these planes and perpendicular to their intersection, their angle is the inclination of these planes, or the diedral angle made by the two faces of the triedral angle whose edge is IO. We therefore say—*A spherical triangle determines a triedral angle whose vertex is at the centre of the sphere, whose plane angles are measured by the sides of the spherical triangle, and whose diedral angles are respectively the angles of the spherical triangle.*

289. It will be readily seen that as the sum of the plane angles in a triedral angle, approaches to four right-angles, the value of each of the diedral angles approaches to two right-angles; and their sum of course approaches to six right-angles; and the less the plane angles of the triedral angle the nearer does the sum of the diedral angles approach to two right-angles. Therefore—*The sum of the three angles of a spherical triangle is not constant; and the limits of this variation are two right-angles and six right-angles.*

*Remark.* The other properties of spherical triangles analogous to those of plane triangles, discussed in articles 38 to 52 inclusive, the learner is requested to investigate by processes analogous to those adopted in the articles referred to.

290. If through the point E the centre of the less circle DGFH, a straight line be drawn perpendicular to this circle, it will pass through the centre of the sphere (282); it is called the *axis* of this circle, and is also the axis of

every circle of the sphere, parallel to this. It will meet the spherical surface in two points which are the *poles* of this circle, and also the poles of all parallel circles in the sphere. Each of these poles is equally distant from every point in the circumference of this circle (204); that is, the chords drawn from the pole of any circle to the several points in its circumference, are all equal; consequently the arcs are equal. This enables us to describe the circumference of a circle upon the surface of a sphere from one of its poles, without finding the centre of the circle. Fig.147.

291. A plane is *tangent* to a sphere when it touches it in only one point, as C (*fig.* 148). This point is called the point of contact, and as every other point of this plane is without the sphere, it must be at a greater distance from its centre (281). Therefore a straight line drawn from the centre of the sphere to the point of contact, must be perpendicular to the tangent plane (205). Hence—*A plane tangent to a sphere is perpendicular to radius at the point of contact.* Fig.148.

292. Let us inscribe in any arc of a semi-circle (*fig.* 149) a portion of a regular polygon, and circumscribe about the same arc the corresponding portion of a regular polygon of the same number of sides. The area generated by the portion $abcd$ of the inscribed polygon, in a revolution of the figure about the diameter $ap$, is composed of several conical surfaces generated by the sides. The side $ab$ generates a complete conical surface; the others, surfaces of truncated cones. The area of one of these surfaces, that generated by $cd$ for instance, is obtained by drawing from $l$ the middle of $cd$, $lq$ perpendicular to $a$O; it is expressed by $(cd) \times circ.\ lq$, (272). This expression may be changed to one more general; for this purpose, draw $cr$ perpendicular to $d$O, and draw also $l$O. The triangles $dcr$, $ql$O, having their sides perpendicular to each other, are equiangular and therefore similar, and give the proportion $\frac{cd}{cr} = \frac{l\text{O}}{lq}$. But as $cr$ is equal to $f$O, and as the circumferences of circles are to each other as their radii, we have $\frac{cd}{f\text{O}} = \frac{circ.\ l\text{O}}{circ.\ lq}$; and, multiplying by the denominators, we obtain Fig.149.

Fig.149. $(c\,d) \times circ.\ l\,q = (f\,O) \times circ.\ l\,O$; consequently the area of the conical surface described by $c\,d$, may have for its expression $(f\,O) \times circ.\ l\,O$; that is, *the product of the height multiplied by the circumference of the circle inscribed in the polygon of which this line is a side.*

As there is nothing in the reasoning which fixes the part of the arc in which this line $c\,d$ is situated, it applies equally to every side of the polygon; and consequently the sum of the areas of the surfaces described by the several sides of the polygon between $a$ and $d$, will be the sum of their heights multiplied by the common factor, the circumference of the inscribed circle; that is, $(a\,O) \times circ.\ l\,O$.

So also the area of the body generated by the revolution of the portion of the circumscribed polygon. will be $(AO) \times circ.\ OL$. If we had continued the polygons quite through the semi-circumference (the arc $a\,b$ being supposed a measure of the semi-circumference), the surface generated by the larger polygon would be equal to the height of the polygon multiplied by the circumference of the circle; and the area of the surface generated by the less polygon, would be equal to its height multiplied by the circumference of the inscribed circle.

These polygons, by having the number of their sides increased, may be made to approach each other so nearly that the difference between the surfaces generated by them respectively, shall be as little as we please. But as these surfaces approach to an equality, the products expressing them approach also to an equality. The height of the inscribed polygon is constantly equal to the diameter of the revolving semicircle; the other factor, the circumference of the inscribed circle, continually increases with the increasing number of the sides of the polygon; and the *limit* of this increase is the circumference of the circle in which the polygon is inscribed. The limit, therefore, of *increase* for the product expressing the area of the surface described by the interior polygon, is that of *the diameter multiplied by the circumference of the circle.*

With respect to the other product, the height of the greater polygon is diminished continually as the number of sides increase; and the limit of this diminution is the diameter of the circle; the other factor remains constantly the circumference of the circle; so

so that the limit of *decrease* for the product express- Fig.149.
ing the area of the surface generated by the exterior polygon, is that of *the diameter multiplied by the circumference of the circle.* But the area of the surface generated by the semicircular arc, is the *limit* to which the areas, generated by the polygons, approach as the number of sides in the polygons are increased. It therefore follows, that the diameter multiplied by the circumference expresses an area neither greater nor less than the area generated by the semicircumference in a revolution about its diameter. This surface is the surface of a sphere (281), which we designate by $S$. We therefore say—*The area of a sphere has for its measure, the product of its diameter multiplied by the circumference of a great circle:* That is, $S = 2R \times circ.$

293. *Remark* 1*st.* The area of a great circle has for its measure, half the radius multiplied by the circumference, or $\frac{1}{2} R \times circ. = \pi . R^2$. Therefore—*The area of a sphere is equivalent to that of four great circles of the sphere.* And we have $S = 4\pi . R^2$.

294. *Remark* 2*d.* It follows from the above reasoning, that any portions of the same spherical surface comprehended between parallel planes, and called *spherical zones, are to each other as the perpendicular distances of the planes by which they are determined;* whether these planes are secants or secants and tangents.

295. The portion of the spherical surface comprehended between the two semi-circumferences ALB, AIB, (*fig.* 147) is called a *lunary surface*, and the angle IAL, Fig.147.
the *lunary angle.* Suppose the angle IAL of this lunary surface to be commensurable with four right-angles, (that is, the sum of all the angles made by arcs diverging from the point A); and suppose that their common measure is contained forty-eight times in four right-angles, and ten times in the angle IAL. If we divide the angle IAL into ten equal parts, and the remaining angular space about the point A, into parts equal to $\frac{1}{10}$ of IAL; and produce the arcs till they meet in the point B; the entire spherical surface will be divided into forty eight equal lunary surfaces, of which the lunary surface BIALB contains ten; therefore this lunary surface is to the entire surface of the sphere, as ten is to forty-eight or as the lunary angle is to four right-angles.

Fg .147. If the lunary angle IAL is incommensurable with four right-angles, we can take a measure of four right-angles so small that the remainder after dividing the angle IAL by this measure, is less than any assignable angle, which remainder we designate by $d$, and the lunary surface answering to this angle by $m$; the lunary angle *minus* this remainder, IAL — $d$, is commensurable with four right-angles; and they therefore gives us the proportion $\frac{spher.\ surf.}{lun.\ surf. - m} = \frac{4\ right\text{-}angles}{\text{IAL} - d}$. Suppose that we apply this small measure of four right-angles, once more than it is contained in the lunary angle, so that the angular space occupied by this measure shall exceed the lunary angle by an indefinitely small excess, which excess we denote by $d'$, and the lunary surface answering to this excess by $m'$. The angular space thus occupied by this small measure, is commensurable with four right-angles, and we shall consequently have the proportion $\frac{spher.\ surf.}{lun. surf. + m'} = \frac{4\ right\text{-}angles}{\text{IAL} + d}$. We see therefore that, if any thing be added to the lunary surface, a corresponding quantity must be added to the lunary angle to balance the proportion; and if any thing be subtracted from the lunary surface, a corresponding quantity must be subtracted from the lunary angle to balance the proportion; that is, the sum of four right-angles is to the lunary angle as the entire spherical surface is to a magnitude neither greater nor less than the lunary surface. And we say—*The area of a lunary surface is to the area of a sphere, as the lunary angle is to four right-angles.*

Representing the lunary surface whose angle is IAL by *Lun.A*, calling $S$ the area of the spherical surface, and denoting a right-angle by $D$; we shall have

$$Lun.A = S \times \frac{\text{IAL}}{4D}.$$

296. ***Remark.*** If two great circles cut each other at right-angles, they will divide the surface of the sphere into *four* lunary surfaces whose angles are all right-angles. And a third great circle perpendicular to the common intersection of the first two circles, will divide each of these lunary surfaces into two triangles, each of whose angles is a right-angle. The surface of the sphere, therefore, is composed of *eight* spherical triangles whose angles are all right-angles.

297. It will be perceived that the three great circles ACBL, CILK and MIFK, which form the spherical triangle CIM, divide the spherical surface into eight triangles, of which CKM and FIL are *symmetrical*, which is readily seen by observing that the triedral angles at the centre, to which they correspond, are equal in all their parts. This equality of the parts of the triedral angles, shows at once the equality of the parts of the corresponding spherical triangles. But this equality of the parts of one spherical triangle, to the parts of another, does not, as in plane triangles, prove the *equality* of the triangles, unless the equal parts are arranged in the same order; their parts being arranged in the contrary order, we cannot turn one of them over as we did the plane triangle (39) to make the two coincide; for placing them so that their vertices may coincide, the two triangles would be on opposite sides of the plane passing through the three points of these vertices; they are therefore symmetrical. Fig.147.

298. Let ABC, DEF, (*fig.* 150) be two symmetrical triangles. Let P be the nearest pole of the less circle whose circumference passes through the vertices A, B, and C; and let Q be the nearest pole of the less circle whose circumference passes through the three vertices of the other triangle. If we draw the equal arcs PA, PB, PC, and also the equal arcs QD, QE, QF; it is evident that the three last arcs will be equal to the three first; for the relative position of the points D, E, F, being symmetrical with that of the three points A, B, C, the curvature of the circumference passing through the first three points, must be the same with the curvature determined by the other three points (136); the circles are therefore equal, and consequently at equal distances from their poles; and the arcs PA, PB, PC, QD, QE, QF, are all equal. We have then three isosceles spherical triangles PAB, PBC, PAC, equal respectively to the three spherical triangles QDE, QEF, QDF; the quadrilateral PABC, composed of the two triangles PAB, PBC, must therefore be equivalent to the quadrilateral QDEF, composed of the two equal triangles QDE, QEF; if then we take from the one quadrilateral the triangle PAC, and from the other, the equal triangle QDF, the remaining triangle ABC of the one will be equivalent to the remain- Fig.150.

Fig.150. ing triangle DEF of the other; and we say—*Two symmetrical spherical triangles are equal in area.*

Fig.147. 299. The two opposite triangles CMK, LFI, (*fig.* 147) being symmetrical, are equal in area; if we add to each of these equal areas the triangle CMI, we shall have CMK $+$ CMI $=$ CMI $+$ LFI; but the triangles CMK, CMI, constitute the lunary surface whose angle is CIM; the sum of two triangles CMI, LFI, in the *nearer hemisphere*, is therefore equivalent to this lunary surface. We have therefore this general rule:—*If two semi-circumferences cut each other in a hemisphere, the sum of two opposite triangles will be equal to a lunary surface whose angle is equal to the angle made by the semi-circumferences.*

Fig.151. 300. Let ABC (*fig.* 151) be a spherical triangle, and DEFGHI the circumference of a great circle which does not cut this triangle. Produce the sides of the triangle both ways till they meet this circumference; and we shall have IBD $+$ GBF $=$ *Lun.*B, HAG $+$ DAE $=$ *Lun.*A, FCE $+$ HCI $=$ *Lun.* C. It is manifest that the sum of these *six* triangles is equal to the hemisphere *plus* twice the triangle ABC, that is, equal to 2 ABC $+$ *Lun.* 2 $D$; we shall therefore have 2 ABC $+$ *Lun.* 2 $D$ $=$ *Lun.* (A $+$ B $+$ C), or ABC $=$ *Lun.*$\frac{1}{2}$(A $+$ B $+$ C $-$ 2 $D$). We say therefore—*The area of a spherical triangle is equivalent to half the lunary surface whose angle is equal to the sum of the three angles of the triangle minus two right-angles.*

Suppose $a$, $b$, $c$, to designate the arcs which measure the angles A, B, C, of any spherical triangle; and let $C$ denote the circumference of a great circle, and $R$ the radius. We shall have, for the area of the spherical triangle, $R \times (a + b + c - \frac{1}{2} C)$.

301. To ascertain the volume of the body generated by the revolution of the polygonal sector $abcd$O (*fig.*

Fig.149. 149), it will be necessary to examine the body generated

Fig.152. by the isosceles triangle ABO (*fig.* 152) revolving about the straight line OD as an axis. Produce AB till it meet the axis in D; and from the points A, B, draw AM and BN, perpendicular to OD.

The volume of the body generated by the triangle AOD, will have for its measure $\frac{1}{3}\pi \times (\text{AM})^2 \times \text{OD}$, (174). The body generated by the triangle OBD, has for its measure $\frac{1}{3}\pi\,(\text{BN})^2 \times \text{OD}$; therefore the difference of these bodies, or the body generated by ABO,

will have for its measure $\frac{1}{3}\pi . [(AM^2) - (BN^2)] \times OD$. Fig.152.
This expression may take another form. From I, the middle of AB, draw IK perpendicular to OD, and through B draw BP parallel to OD; we shall have $AM + BN = 2\ IK$, and $AM - BN = AP$; consequently $(AM + BN) \times (AM - BN) = (AM)^2 - (BN)^2$, (176), $= 2\,IK \times AP$. The volume of the body generated by ABO, may therefore be expressed by $\frac{2}{3}\pi \times IK \times AP \times OD$. But if we draw OI, as OAB is an isosceles triangle, OI will be perpendicular to AB; and the two triangles APB, OID, are similar, and give the proportion $\frac{OI}{AP} = \frac{OD}{AB}$; hence $AP \times OD = OI \times AB$; and substituting $OI \times AB$ for its value in the above expression we have $\frac{2}{3}\pi \times IK \times AB \times OI$. But the similar triangles ABP, OIK, give the proportion $\frac{AB}{BP} = \frac{OI}{IK}$, wherefore $AB \times IK = OI \times BP = OI \times MN$; and substituting this value of $AB \times IK$ in the expression for the volume of the body in question, we have $\frac{2}{3}\pi \times OI^2 \times MN$. The area of a circle whose radius is OI is expressed by $\pi \times (OI)^2$. We therefore say—*The volume of a body generated by the revolution of an isosceles triangle about a straight line which meets it only in its summit, and is in the same plane, has for its measure two thirds of the area of a circle whose radius is the height of the triangle, multiplied by the part of the axis intercepted by perpendiculars drawn to it from the vertices at the base of the triangle.*

**302.** [If the axis of revolution is parallel to the base of the triangle, MN will equal AB, the surface generated by AB will be a cylindrical surface, and the volume of the body generated by the triangle will be *two thirds of the volume of a cylinder* the radius of whose base is OI, the height of the triangle, and whose height is AB, the base of the triangle. It follows, therefore, that the volume of a body generated by the revolution of the same triangle about its base as an axis, will be equivalent to *one third of the cylinder* of the above dimensions.]

**303.** Returning to figure 149, it will be readily seen that the portion of the polygon *abcd*, inscribed in the quadrant *ad* is composed of isosceles triangles whose common height is the radius of the inscribed circle. The volume of the body generated by *aOb*, will have for its

Fig.149. measure $\frac{2}{3}\pi \times (Oh)^2 \times ae$; that generated by $bOc$ will have $\frac{2}{3}\pi \times (Oi)^2 \times ef$; and that generated by $dOc$ will have $\frac{2}{3}\pi \times (Ol)^2 \times fO$. As $Oh$, $Oi$ and $Ol$ are equal, the sum of these bodies, or the body generated by the portion of the inscribed polygon $abcd$, will have for the meaaure of its volume, $\frac{2}{3}\pi \times (Oh)^2 \times (ae + ef + fO)$, or $\frac{2}{3}\pi \times (Oh)^2 \times aO$; and if the portion of the inscribed polygon occupied the entire semicircle, the body generated by it, in a revolution about the diameter $ap$, will have for its measure $\frac{2}{3}\pi \times (Oh)^2 \times ap$.

If we apply this result to the body generated by the circumscribed polygon, we shall have for the measure of its volume, $\frac{2}{3}\pi \times (aO)^2 \times AP$.

By increasing the number of sides of the polygons, the inscribed polygon increases continually, and the limit of this increase is the semicircle; but with this increase of the number of sides, the factor $Oh$ increases, and the limit of this increase is $Oa$; and as the other factors remain the same, the limit for the product is $\frac{2}{3}\pi \times (Oa)^2 \times ap$.

But by increasing the number of sides of the exterior polygon, it becomes continually less, and the limit of this diminution is the semicircle; but with this increase in the number of sides, the factor AP is continually diminished, and the limit of this diminution is $ap$; and as the other factors remain the same, the limit of the product expressing the volume of the body generated by the portion of the polygon circumscribed about the semicircle, is $\frac{2}{3}\pi \times (Oa)^2 \times ap$. It appears therefore that $\frac{2}{3}\pi \times (Oa)^2 \times ap$, cannot be the measure of a volume greater or less than that of the body generated by the semicircle whose radius is $aO$, revolving about the diameter $ap$; but this body is the sphere whose radius is $aO$.

If we denote $aO$ by $R$ and $ap$ by $2R$, the expression becomes $\frac{2}{3}\pi \times R^2 \times 2R$, or $\frac{4}{3}\pi \times R^3$; and may take the form $4\pi . R^2 \times \frac{1}{3}R$. But $4\pi R^2$ is the area of the sphere whose radius is $R$; we therefore say—*The sphere has for the measure of its volume, the area of its surface, multiplied by one third of its radius*; or *the area of a great circle multiplied by two thirds of the diameter.*

304. If we have a cone and a cylinder whose common Fig.148. height is equal to the diameter of the sphere (*fig.* 148),

and whose common base is equal to a great circle of the sphere; we may express these convex surfaces as follows: Fig 148.

$$\text{that of the cone} = (5)^{\frac{1}{2}}\,\pi.R^2;$$
$$\text{sphere} = 4\,\pi.R^2;$$
$$\text{cylinder} = 4\,\pi.R^2.$$

The volumes of these bodies have the following expressions:

$$\text{that of the cone} = \tfrac{2}{3}\,\pi \times R^3 = \tfrac{1}{3}\,\pi.R^2 \times R;$$
$$\text{sphere} = \tfrac{4}{3}\,\pi \times R^3 = \tfrac{2}{3}\,\pi.R^2 \times 2R;$$
$$\text{cylinder} = \tfrac{6}{3}\,\pi \times R^3 = \tfrac{2}{3}\,\pi.R^2 \times 3R.$$

*Remark.* The sphere is therefore *two thirds* of the circumscribed cylinder. The cone is *one third* of the circumscribed cylinder. The *area* of the sphere is equivalent to that of the convex surface of the circumscribed cylinder.

305. The bodies of revolution which have now been discussed, are said to be *similar* when the figures which generate them are similar. Two cones generated by similar triangles revolving about homologous sides, have their heights in proportion to the radii of their bases.

Let $A$ represent the area of a cone whose height is $H$ and the radius of whose base is $R$, and whose side is $S$; we shall have the equation $A = 2\pi.R \times \frac{1}{2} S$, (272). For another similar cone we shall have $a = 2\pi.r \times \frac{1}{2} s$. If we compare these expressions, cancelling the common factors, we have $\frac{A}{a} = \frac{R \times S}{r \times s}$; or because $\frac{R}{r} = \frac{S}{s}$, we may write $\frac{A}{a} = \frac{R^2}{r^2} = \frac{S^2}{s^2} = \frac{H^2}{h^2}$. We therefore say—

*The areas of similar cones are to each other as the second powers of their corresponding dimensions.*

306. Two similar cylinders are generated by similar rectangles (*fig.* 144). Their areas will have for their expressions, $A = 2\pi.R \times H$, $a = 2\pi.r \times h$. If we compare these neglecting the common factors in the terms of the second ratio, we shall have $\frac{A}{a} = \frac{R \times H}{r \times h} = \frac{R^2}{r^2} = \frac{H^2}{h^2}$; Fig.144.

that is—*The areas of similar cylinders are to each other as the second powers of their homologous measures.*

307. All spheres are similar, because every sphere has for its *generatrix* a semicircle, and all semicircles

are similar. The areas of two spheres have for their expressions, $A = 4\pi . R^2$, $a = 4\pi . r^2$; comparing these and cancelling the common factors in the second ratio, will give $\frac{A}{a} = \frac{R^2}{r^2}$; that is—*The areas of two spheres are to each other as the second powers of their radii.*

308. If we compare the *volumes* of two similar cones, which are expressed as follows; $V = \frac{1}{3}\pi . R^2 \times H$, and $V = \frac{1}{3}\pi . r^2 \times h$; we shall have $\frac{V}{v} = \frac{R^2 \times H}{r^2 \times h} = \frac{R^3}{r^3} = \frac{H^3}{h^3}$ as, $\frac{R}{r} = \frac{H}{h}$; that is—*The volumes of similar cones are to each other as the third powers of their homologous lines.*

309. The volumes of two similar cylinders have for their expressions, $V' = \pi . R^2 \times H$, $v' = \pi . r^2 \times h$. If we compare these volumes, we have $\frac{V'}{v'} = \frac{R^2 \times H}{r^2 \times h} = \frac{R^3}{r^3} = \frac{H^3}{h^3}$, because $\frac{R}{r} = \frac{H}{h}$. Whence we say—*The volumes of similar cylinders are to each other as the third powers of their homologous measures.*

310. Two spheres, being always similar bodies, and having for the expressions of their volumes, $V'' = \frac{4}{3}\pi . R^3$, and $v'' = \frac{4}{3}\pi . r^3$, may be readily compared; and will give $\frac{V''}{v''} = \frac{R^3}{r^3} = \frac{D^3}{d^3}$. That is—*The volumes of two spheres are to each other as the third powers of their radii, or as the third powers of their diameters.*

311. It was observed (227) that a regular polyedron has all its faces equal regular polygons, all its diedral angles equal, and all its polyedral angles equal.

Let us inquire into the *number* of regular polyedrons which may be constructed.

In the first place, it is manifest that the faces of these regular bodies must be equilateral triangles, squares or regular pentagons; for the angles of a regular hexagon are equal to four right angles; no polyedral angle therefore, can be formed with regular hexagons, or with regular polygons of a greater number of sides.

Of those bodies contained by equilateral triangles, the

polyedral angles must be either triedral, tetraedral or pentaedral; for six angles of 60° each are equal to four right-angles, and cannot therefore form a convex polyedral angle.

The *regular tetraedron* is contained by *four* equilateral triangles; and has four triedral angles. Triedral angles admit only of four faces in the body. Fig.153.

The *regular octaedron* is contained by *eight* equilateral triangles; and it has six tetraedral angles. The tetraedral angle allows only eight faces in the body. Fig.154.

The *regular icosaedron*, contained by *twenty* equilateral triangles, will have twelves pentaedral angles. These are all the regular polyedrons which can be formed with triangular faces. Fig.155.

The *regular hexaedron* or *cube* is evidently the only regular polyedron which can be formed by square faces. Fig.156.

The *regular dodecaedron* contained by twelve pentagonal faces, has twenty triedral angles. Fig.157.

These *five* bodies are the only regular polyedrons. They can each of them be inscribed in a sphere, and each may have a sphere inscribed in it; and, in each case, the inscribed and circumscribed sphere will have the same centre. The right cone whose angle at the apex is 60°, and the right cylinder whose height is equal to the diameter of its base, may each have an inscribed and circumscribed sphere whose centres are the same point. The right prism and pyramid, whose heights have the proper ratio to their other dimensions, and whose bases are regular polygons, may have inscribed and circumscribed spheres with a common centre.

Besides the five polyedrons mentioned above, the *sphere* is the only other *regular body*.

## *Practical Questions.*

What is the volume of a prism whose height is 20 feet and whose base is a rectangle the contiguous sides of which are 3 feet and 8 feet?

Required the cubic feet in a stick of timber 60 feet long, 18 inches wide and 8 inches thick?

What is the volume of a prism whose height is 7 yards, and whose base is a right-angled triangle one side of which is one foot and the other 30 inches?

Required the volume of a prism whose height is 12 feet and whose base is an equilateral triangle whose side is 6 feet.

A rectangular parallelopiped has its three contiguous edges, 4 feet, 6 feet, and $3\frac{1}{2}$ feet; what is it its volume?

An obelisk is composed of the frustum of a square pyramid, the side of whose inferior base is 12 feet, the side of the superior base 6 feet and the height $68\frac{1}{2}$ feet, this is surmounted by a pyramidal summit $1\frac{1}{2}$ feet high, making the entire height 70 feet; what is the entire volume of the obelisk?

Suppose the above obelisk to be of stone, what would it cost at one dollar per foot of surface, the base of the obelisk not being included?

What is the area of a rectangular parallelopiped whose three contiguous edges are 3 feet, 4 feet, and 5 feet?

What is the area of a right pyramid, the base of which is a square of 6 feet on a side, and whose height is 4 feet?

A house 40 feet wide and 50 feet long, has a cellar under the whole of it, 8 feet deep below the sills; the bottom of the sills (which are 10 inches wide) is 2 feet above the natural surface of the ground; the inside of the cellar wall, which is 22 inches thick, coincides with the inside of the sills; the underpinning stones are 18 inchess deep and 8 inches thick; there are 6 cellar windows of 3 feet each in width and 18 inches high. How many cubic yards of earth were removed from the cellar? What did the cellar wall cost at $3 per cubic yard? How many cubic feet of stone in the underpinning; and what did it cost at 50 cents per foot of surface?

A ditch is to be dug 1 mile in length, 5 feet in width

and 4 feet in depth; what will it cost at 10 cents per cubic yard?

An embankment for a road is to be made across a meadow, $\frac{1}{4}$ of a mile in width; the road is to be 6 feet above the surface of the meadow, 30 feet wide at top, with banks sloping 2 feet in 3; what will this embankment cost at $12\frac{1}{2}$ cents per cubic yard?

A bridge is to be built of stone; its length 60 feet, its width 25 feet and its height 12 feet, with two semi circular arches of 8 feet radius and 9 feet high in the intrados; What will be the cost of this bridge, at $2 per cubic yard?

What is the area of a right cone the radius of whose base is 36 feet, and whose height is 48 feet? What is the volume of this cone?

What is the area of a cylinder whose height is 20 feet and the radius of whose base is 10 feet? What is its volume?

What is the area of a sphere inscribed in the same cylinder; and what is its volume?

What are the area and volume of a right cone inscribed in this cylinder?

How many cubic feet of water will a hemispherical vessel contain, its radius being 10 feet?

How many cubic feet of water in a pond of 1000 acres, 10 feet deep? How many miles of canal will such a reservoir supply for three months, at 50 cubic feet per mile per minute; supposing one foot in depth of the reservoir to be lost by evaporation?

Supposing this reservoir is at one extremity of the canal to be supplied by it, what will be the velocity of the current in this part of the canal, with 5 feet of water and an average width of 40 feet?

A cubic block measures 6 feet on a side; what is the edge of a cubic block eight times as large?

*Cleopatra's Needle*, a celebrated pyramidal obelisk near Alexandria in Egypt, consists of a single piece of red granite; its height is 60 feet, and its inferior base is 7 feet square. Supposing its superior base to be 5 feet square, what will be its area and volume; and what its weight at 170 lbs per cubic foot?

The pyramid of Cheops, the largest of the Egyptian pyramids, has a square base whose side is 693 feet; its height is 499 feet. How many acres of ground does it

cover; and what is the side of a cube of equal volume?

Suppose 1000 men to be employed at a time, in erecting this structure, and suppose 10 men, on an average, to complete 29 cubic feet of the work per day: How long would it be in building?

# AN INTRODUCTION

TO

# DESCRIPTIVE GEOMETRY.

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## SECTION I.—*Preliminary Explanations.*

1. IN the Second Part of the Elements of Geometry we discussed geometrical magnitudes embracing the *three dimensions* of space. But it must have been perceived by the learner, that, in the graphical representations of these bodies, there was much indefiniteness respecting the relative position and relative measure of their several parts. In the application of Geometry to the arts, we have frequent occasion, not only to represent the various forms and relative positions of bodies, but accurately to determine the various dimensions and other relations of their parts, and to deduce from these determinate relations, a series of results, the accuracy of which must depend entirely upon the fidelity of the graphical representation of the given magnitudes, and of the graphical *construction* of the consequent relations.

2. The object of Descriptive Geometry is to teach this accurate representation of bodies upon a plane, and this graphical solution of problems. A geometrical magnitude is represented upon a plane by a method called *projection.*

The simplest projection, and that in most frequent use, is called *orthogonal* or *othographic projection.* It consists in drawing straight lines from the points to be projected, perpendicular to the plane on which the projection is to be made, and which is called *the plane of projection;* the straight lines by which the projection is made are called *projecting lines.*

3. When the projecting lines are inclined to the plane of projection, the projection is called *oblique.* When the

projecting lines converge towards some fixed point, the projection is called *linear perspective.*

The orthographic projection may be considered a perspective in which the point of convergence is at an infinite distance from the body.

4. In the orthographic projections used for the general purposes of Descriptive Geometry, two *planes of projection* are commonly used; and these are usually taken perpendicular to each other, the one horizontal and the other vertical, as the construction is thus rendered the most simple.

These planes are called *co-ordinate planes.* They are considered to be indefinitely extended; and will therefore meet. The *line of intersection* is horizontal, and is called the *ground line.*

Fig. 1. Let AB (*fig.* 1) represent the ground line, and the plane of the paper the horizontal plane of projection. The vertical plane of projection will pass through AB perpendicular to the plane of the paper. We may conceive this vertical plane to revolve about the ground line as an axis, till it shall coincide with the other plane, that is, the plane of the paper, the part of the vertical plane *above* the horizontal moving backward to coincide with that part of the horizontal plane which is *behind* the vertical. That part of the plane of the paper which lies above the ground line, will now represent not only that portion of the horizontal plane which lies *behind* the vertical, but also that portion of the vertical plane which is *above* the horizontal; and that part of the plane of the paper which lies below the ground line, will represent that portion of the horizontal plane which is *in front* of the vertical, and also that portion of the vertical plane which is *below* the horizontal.

5. This rectangular position of the co-ordinate planes gives us *four* diedral angles, each of which is a right angle (El. 208).* The diedral angle in front of the vertical plane and above the horizontal, we call the *first angle;* that behind the vertical and above the horizontal, we call the *second;* the diedral angle behind the vertical and below the horizontal, the *third;* and that below the horizontal and in front of the vertical, the *fourth.*

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* This reference is to the *Elements*, article 208.

In the projection of geometrical magnitudes generally, the planes of projection may be so chosen that the points to be projected shall be all in the first diedral angle. The projection consists of two parts. First, the several points of the body are projected by perpendiculars drawn from them to the horizontal plane; the intersections of these perpendiculars with the horizontal plane, make their projection upon this plane: This is called their *horizontal projection.* Secondly, the several points are projected by perpendiculars drawn to the vertical plane: the intersections of these perpendiculars with the vertical plane, constitute the *vertical projection* of these points. Fig. 1.

6. This projection is used by architects and engineers in representing the several parts of any edifice or structure. A drawing is made upon paper, which is a miniature representation of a horizontal projection of the edifice. This is called the *ground plan*, or simply the *plan* of the edifice; and exhibits the relative situations of all the remarkable points in the edifice, referred to a horizontal plane by perpendiculars to this plane.

The *plan* of an edifice, it will be perceived, shows us nothing respecting the height of its different parts. In order to fix the heights, and therefore to determine the relative positions of the several points in the edifice, as well as its various dimensions, a drawing is also made representing a vertical projection of these points.

When the vertical projection represents the interior of the building, it is called a *section* or *profile*. When this projection exhibits the exterior of the building, it is called an *elevation*.

It is manifest that those lines in the building which are oblique to each of the planes of projection, will not be projected in their true magnitudes upon either of the planes. We shall see, however, that descriptive geometry accurately determines these magnitudes notwithstanding this apparent defect of the process.

7. Let M (*fig.* 1) be a point in the first angle. Draw MM′ perpendicular to the horizonal plane, and MM″ perpendicular to the vertical plane; M′ is the horizontal projection of the point M, and M″ is the vertical projection of this point. If we suppose a plane to pass through

Fig. 1. the projecting lines MM′, MM″, it will be perpendicular to each of the planes of projection BC, BD, (El. 207), and therefore perpendicular to the ground line (El. 209). But if *M*M″ be perpendicular to AB, by revolving the vertical plane, about the ground line, through the second diedral angle into the plane of the paper, *M*M″ will be in the same straight line with *M*M′ also perpendicular to AB (El. 14). We therefore say—*The vertical and horizontal projections of the same point, are in a straight line perpendicular to the ground line.*

8. *Remark.* The distance of the vertical projection of a point from the ground line is equal to the distance of the point itself from the horizontal plane; and the distance of the horizontal projection of a point from the ground line, is equal to the distance of the point, in space, from the vertical plane.

It follows, therefore, that *all points situated in the vertical plane, have their horizontal projections in the ground line; and all points situated in the horizontal plane, have their vertical projections in the ground line. All points in either plane are their own projections in that plane.*

9. *The two projections of a point determine its position;* for M′ and M″ being the two projections of a point, if we draw through the projections straight lines perpendicular to each of these planes, these perpendiculars must pass through the proposed point; it must therefore be at their intersection M.

When we speak of a point in space given in position by its projections, we say, the point (M′, M″); by which we mean, the point whose horizontal projection is M′ and vertical projection M″.

10. Two lines which are parallel or which cut each other, determine the position of a plane (El. 197). *If, therefore, we have the intersections of any plane with the two planes of projection, this plane is given in position.*

11. The two intersections of the proposed plane with the planes of projection, are called its *traces* upon these planes. Its intersection with the vertical plane is called its *vertical trace;* and its intersection with the horizontal plane, is its *horizontal trace.* If the proposed plane be parallel to one of the planes of projection it can

have no trace upon that plane. Therefore, if a plane has but one trace, it is parallel to that plane of projection which contains no trace. A plane whose horizontal trace is M'N', and whose vertical trace M''N'', is called usually, the plane (M'N', M''N''). Fig. 1.

12. If a plane be inclined to each of the planes of projection, but parallel to the ground line, it can never meet the ground line (El. 198); and consequently its traces upon the planes of projection cannot meet the ground line; they are therefore parallel to it: And we say—*A plane parallel to the ground line, will have its traces parallel to the ground line. And the traces of a plane inclined to this line, will intersect it.*

13. If the two traces of the proposed plane are perpendicular to the ground line, the plane itself must be perpendicular to the ground line (El. 199); and if the ground line is perpendicular to the proposed plane, the co-ordinate planes, both passing through this line, must be perpendicular to the proposed plane (El. 207). Therefore—*If the two traces of the proposed plane are perpendicular to the ground line, the plane itself is perpendicular to each of the co-ordinate planes.*

14. As one point is projected by a perpendicular drawn to the plane of projection; all the points in a straight line are projected by perpendiculars drawn to the plane of projection (*fig.* 2). These perpendiculars Fig. 2. being parallel, and passing through the same straight line, are in the same plane. We therefore say—*The horizontal projection of a straight line, is the horizontal trace of a plane passing through the line, and perpendicular to the horizontal plane. The vertical projection of a straight line, is the vertical trace of a plane passing through the line, and perpendicular to the vertical plane.* These planes are called *projecting planes.**

15. It is here supposed that the proposed line is not perpendicular to either of the co-ordinate planes. *The projection of a straight line perpendicular to the plane of projection, is a point.*

16. *If a line be parallel to either of the co-ordinate*

* It is important to distinguish between *projecting planes* and the *planes of projection.*

Fig. 2. *planes, its projection upon the other plane will be parallel to the ground line.* For the line itself and the projecting line of any point in it, are two lines which determine this projecting plane to be parallel to one of the planes of projection (El. 197); their intersections with the other plane of projection are therefore parallel (El. 214).

From this it follows that—*The projections, on the same plane, of parallel lines, are parallel.*

If, however, the lines are perpendicular to either of the co-ordinate planes, their projections will be points, and cannot be said to be parallel.

17. *Two projections of a line determine its position in space.* For drawing a plane through its horizontal projection perpendicular to the horizontal plane, the plane must pass through the proposed line; and passing a plane through the vertical projection perpendicular to the vertical plane, this plane will also pass through the line; the line must therefore be the intersection of these two planes, which is determined, the planes themselves being determined: The line is therefore *given in position by its projections.*

*Remark.* It is readily seen that each of the projections of the proposed line, answers to an infinite number of lines all in this projecting plane; but the two projections answer only to a line in each of the planes; this line must be their intersection.

If the proposed line were in a plane perpendicular to each of the co-ordinate planes, and inclined to these planes, the two projections would not determine it. Some other condition would be necessary; such as its projection upon a third plane, or the distances of two points of the line from their projection on one of the planes.

When we have occasion to speak of a line given by its horizontal projection M′N′, and its vertical projection M″N″, we say, the line (M′N′, M″N″).

18. [Right lines and planes are considered as indefinite in extent. In the diagrams, those parts of the given or the required lines which are in the first angle, are made full; the *dotted lines* in the figures represent portions of the proposed lines which are situated behind a plane, and supposed to be seen through the plane. Those

lines which are composed of short lines and dots are the *auxiliary lines* used in *constructing the solution.*]

## Section II.—*Of the Straight Line and the Plane.*

19. Problem. *The two projections of a line being given, to determine the points in which it* pierces *the co-ordinate planes.*

Let N′*N* (*fig.* 3) be the horizontal projection of the proposed line, and *MM*″ its vertical projection. The point where it *pierces* the vertical plane must have its horizontal projection in the ground line (8); *N* must therefore be the horizontal projection of this point. If then we draw, in the vertical plane, *N*P″ perpendicular to the ground line, and meeting the vertical projection in P″, this point P″ will be the point in which the proposed line meets the vertical plane. And if we draw in the horizontal plane, the line *M*P′ perpendicular to the ground line, the point P′ will be the point in which the proposed line pierces the horizontal plane. Fig. 3.

It is evident that this problem admits of four cases; as the part of the proposed line between the points where it meets the two planes, may be situated in either of the four diedral angles. The solution in figure 3, answers to the case in which this portion of the line is in the first angle. Figure 4 gives us the second case; namely, that in which the portion of the line between the two points in which it pierces the planes, is in the second angle. The other cases we shall leave to the ingenuity of the learner. Fig. 4.

20. Problem. *The projections of a line being given, to find the angle which it makes with one of the planes of projection.*

The inclination which a line has to a plane, is measured in a plane passing through the line, perpendicular to the first plane (El. 207). Suppose that the angle required, is that which the proposed line makes with the horizontal plane (*fig.* 5). This angle will be in the projecting plane, and will be contained between the line itself and its horizontal projection. If, therefore, the projecting plane be conceived to revolve round its horizontal trace, till it coincide with the horizontal plane, the required angle will, by this process, be transferred to the horizontal plane, and can be measured. Fig. 5.

Fig. 5. For this purpose the horizontal projection P′*N* is considered to be the axis about which the projecting plane P′P″*N* revolves. In such a rotation, it is perfectly manifest that every point in the revolving plane will be, throughout its revolution, at the same distance from the axis; and that any line in the revolving plane, perpendicular to the axis, will be perpendicular to the axis at the end of this revolution. The line *N*P″ will therefore be perpendicular to the axis when the projecting plane shall be revolved into the horizontal plane. If, therefore, we draw *Np* perpendicular to P′*N*, take *Np* equal *N*P″, and draw P′*p*, the angle *p* P′*N* will be the angle required.

21. Problem. *Two points being given by their projections, to find their distance asunder in space.*

Let the two points be (M′, M″) (O′, O″) (*fig.* 5). It is obvious from the preceding process, that *m o* will be the distance required. This might be found at once by drawing through the point M′, M′*m* perpendicular to M′O′, and equal M″*M*; and through the point O′, the line O′*o* perpendicular to M′O′ and equal to O″*O*; this gives us *m o*.

22. *Remark. The projection of a line upon either plane, is less than the line itself, unless the line be parallel to this plane.* The length of a line is equal to the hypothenuse of a right-angled triangle, of which one side is its horizontal projection, and the other, the difference of the heights of the two extremities of the line above the horizontal plane. *The projection of a line parallel to the plane of projection is equal to the line.*

23. Problem. *Through a given point to draw a line parallel to another line given in position by its projections.*

The projections, upon the same plane, of parallel lines are parallel (16); therefore, if through the projection of the given point in each plane, we draw lines parallel to the projection, in that plane, of the given line, these will be the projections of the required line.

24. Problem. *Having the projections of two lines, to ascertain whether they cut each other.*

The horizontal projections of the two lines may inter-

sect, and also their vertical projections, and yet the two lines may not come near each other. If the lines intersect in space they will have one point common; and if they are not in the same plane perpendicular to one of the co-ordinate planes, their projections on each plane will intersect; and these two intersections must be the two projections of the point where the lines actually cut each other. The two projections of the same point are in a straight line perpendicular to the ground line (7); therefore, to ascertain whether two lines given by their projections, actually cut each other, draw from the point where their projections intersect, perpendiculars to the ground line; if these perpendiculars meet the ground line in the same point, the two points where the projections intersect are the projections of the same identical point; and this point must be in each of the given lines; that is, the two given lines cut each other, and are therefore in the same plane.

Figure 6 shows a case in which the projections of the two lines in each of the planes, cut each other, when the lines themselves do not meet. Figure 7 gives us the projections of two lines which actually intersect in space. Fig. 6. Fig. 7.

25. *Remark.* If the two proposed lines cut each other in a plane perpendicular to one of the co-ordinate planes, their projection upon this plane will be but one straight line; if they cut each other in a plane perpendicular to each of the co-ordinate planes, their vertical and horizontal projections will be in one and the same straight line perpendicular to the ground line.

26. Problem. *Two planes being given by their traces, to find the projections of their line of intersection.*

When the traces of the proposed planes upon one of the co-ordinate planes, cut each other, their point of meeting is common to the two planes proposed, and is therefore a point in the required line.

Let (*N*N′, *N*N″), (*M*M′, *M*M″), be the two given planes; it is manifest that the point P′ is common to the two proposed planes and the horizontal plane ABC; this is therefore one of the points sought. The point Q″ evidently belongs to the common section of the two proposed planes with the vertical plane; this then is also a point in the required line. The problem is therefore re- Fig. 8.

Fig. 8. duced to finding the projections of a line passing through the points P′, Q″. The horizontal projection of Q″ is *Q* (8); P′ is its own horizontal projection; P′*Q* is therefore the horizontal projection of the proposed line. For similar reasons *P*Q″ is its vertical projection.

Fig. 9. 27. *Remark* 1*st*. If the traces *N*N′, *M*M′, (*fig*. 9) of the proposed planes upon one of the co-ordinate planes, the horizontal for example, are parallel to each other, the intersection of the proposed planes will be parallel to the horizontal plane; and of this line one point is known, namely, the point P″. As this line is in the same plane with each of the horizontal traces of the proposed planes, it will meet them unless it is parallel to them; but it cannot meet them, unless they meet each other, which is impossible as they are parallel; this line of intersection is therefore parallel to each of the horizontal traces of the proposed planes. This question therefore refers itself to the problem in article 23.

28. *Remark* 2*d*. It may happen that the line of intersection of the proposed planes meets neither of the co-ordinate planes; in this case it must be parallel to them

Fig. 10. and parallel to the ground line (*fig*. 10). To find, under these circumstances, the line of intersection of the proposed planes, it will be necessary to refer to a third plane; which, for the sake of simplicity, we take perpendicular to each of the other two co-ordinate planes. We shall not now detain the reader with this process; a little familiarity with constructions will render the solution perfectly easy.

29. If, however, the proposed planes are parallel to each other, their traces upon each of the co-ordinate planes will be parallel; but it is perfectly manifest that they cannot intersect.

30. Problem. *To find the projections of a point, when we know three planes in each of which it is situated.*

As these three planes have only this single point in common; to find the projections of this point, seek first the two projections of the common section of any two of the proposed planes; this line being cut by the third plane, will thus give the point required.

We shall come to the same result, by finding the intersection of two of the planes, and then seeking the intersection of one of these with the third. These two

intersections will be two straight lines which cut each other. This problem will then refer itself to the preceding (26).

31. The most simple method of determining a point by means of three planes, is to suppose them perpendicular to each other; and then give the distances of the proposed point from three planes parallel respectively to the three planes in which this point is situated.

Suppose the three planes BAC, BAD, and DAC, (*fig.* 11) to be perpendicular to each other, and that we know that the point M in space is situated at a distance MM′ from the first, MM″ from the second, and MM‴ from the third. As parallel planes are equally distant throughout their whole extent (El. 212), if we draw the planes MM″M‴, MM′ M‴, MM′ M″, at the given distances respectively from the three planes BAC, BAD, DAC, and respectively parallel to them, the proposed point will be found at their mutual intersection. Fig. 11.

These three planes, with the co-ordinate planes, form the rectangular parallelopiped whose diagonal is AM. The square of this diagonal is equal to the sum of the squares of the three edges which meet at the proposed point (El. 245); that is $(AM)^2 = (MM')^2 + (MM'')^2 + (MM''')^2$. Whence we say—*The square of the distance of any point in space from the point of meeting of three* rectangular *co-ordinate planes, is equal to the sum of the squares of the distances of the proposed point from each of these planes.*

32. *Remark.* It will be perceived that three planes which cut each other form *eight* triedral angles, in each of which the proposed point may be situated conformably with the above conditions; so that to fix the position in space, of the proposed point, we must not only have its distance from each of the three co-ordinate planes, but must know on which side of each it is situated; or we must in some way particularize the triedral angle in which it is situated.

When a point is given by a line and a plane, it is the same thing as if it were given by three planes; for, instead of the given line we must use its two projecting planes.

33. PROBLEM. *A plane being given by its traces, and a line, inclined to this plane, being given by its projections, to find the point at which the line pierces the plane.*

Find the intersection of the proposed plane with one of the projecting planes of the given line; this line of intersection, being in each of these planes, will meet the given line at the point at which it pierces the proposed plane.

The several steps of this solution may be performed from what was said in article 26. Let (N″*O*, M′*O*)
Fig. 12. (*fig.* 12) be the given plane, and (*Q*Q″, *N*M′) the given line; N″*N*M′ will be the plane which projects this line upon the horizontal plane: M′ and N″ are two points in the common section of this plane with the given plane, N″*M* is the vertical projection of this common section, and P″ is the vertical projection of the point of meeting of this common section with the proposed line; that is, P″ is the vertical projection of the point at which the given line pierces the given plane. If then we draw P″P′ perpendicular to the ground line, and meeting the horizontal projection of the given line, in the point P′, this point will be the horizontal projection of the point sought.

34. Problem. *Having the vertical and horizontal traces of a plane, to find, for any point in the horizontal plane, the height of the corresponding point in the proposed plane.*
Fig. 13. Let (*G*N″, *G*G′) (*fig.* 13) be the given plane, and M′ the given point in the horizontal plane; through M′ pass a vertical plane whose horizontal trace shall be parallel to that of the given plane; the proposed point will be in this auxiliary plane, and therefore in its intersection with the given plane, which intersection will be parallel to the horizontal plane (27); the point N″ will be a point in this intersection, and consequently the line N″*N* will measure the height of the required point above the horizontal plane. M″ is the vertical projection of this point, and M′ its horizontal projection (7).

35. *To find the angle which the inclined plane* (*G*G′, *G*N″) *makes with the horizontal plane* BAC; we suppose, from the required point M in the inclined plane, and from its projection M′ in the horizontal plane, perpendiculars drawn to the intersection *G*G′ of these planes; these will form the right-angled triangle MM′G′, of which we know the two sides MM′, M′G′, and which may therefore be readily constructed. The angle MG′M′ will be the inclination required.

36. Having the horizontal trace of the inclined plane, and the angle of its inclination to the horizontal plane, we may find the height MM′ of the point M above the horizontal plane. Through the point M′ draw *N*M′ parallel to *G*G′ the horizontal trace of the given plane; draw M′G′ perpendicular to *G*G′, and make the angle M′G′M equal to the given angle; this will give us MM′ the height of the point M. Fig. 13.

37. Having the horizontal trace of the inclined plane, and its inclination to the horizontal plane, to find the vertical trace. Find the height of this plane above the point M′, by the last article; and through the point *N*, where M′*N*, parallel to the horizontal trace, meets the vertical plane, draw *N*N″ perpendicular to the ground line; and, taking *N*N″ equal M′M, draw *G*N″, which will be the vertical trace of the proposed plane.

38. Problem. *Through a given point to draw a plane parallel to a given plane.*

We know that the traces of the required plane will be parallel respectively to those of the given plane (29); they are therefore easily drawn, if we can find upon either of the co-ordinate planes one point of the required plane. Suppose, through the proposed point, a straight line parallel to the horizontal trace of the required plane; it will lie wholly in this plane, and will therefore meet the vertical plane in the vertical trace of the required plane.

Let (*M*M′, *M*M″) (*fig.* 14) be the given plane, and (P′, P″) the given point. Through P′ draw P′*E* parallel to M′*M*; draw *E*E″ perpendicular to the ground line, making *E*E″ equal to *P*P″; and draw *N*E″ parallel to *M*M″, and *N*N′ parallel to *M*M′. These two lines *N*N′, *N*N″, are the traces of a plane passing through P (given by its projections, P′, P″) parallel to the given plane (*M*M′, *M*M″). Fig. 14.

39. Let (*M*M′, *M*M″) (*fig.* 15) be an inclined plane, to which the line L′M is supposed perpendicular. Suppose a vertical plane L′EM′ to pass through this line cutting the inclined plane in EM′. This auxiliary plane, being perpendicular to the inclined plane and to the horizontal plane, must be perpendicular to their intersection, that is, to the horizontal trace of the given plane. The horizontal projection of the line L′M, must Fig. 15.

Fig. 15. be the line L′M′; but *M*M′ being perpendicular to the plane L′EM′, is perpendicular to L′M′ (El. 202); that is, the horizontal projection of the given line is perpendicular to the horizontal trace of the given plane.

By a similar process we may show that the vertical projection of the given line is perpendicular to the vertical trace of the given plane: We therefore say—*If a line is perpendicular to a plane, the projections of this line will be respectively perpendicular to the traces of the plane.*

40. The converse of this proposition is also evident; that is—*If the projections of a line are perpendicular respectively to the traces of a plane, the line is perpendicular to the plane.* For the projecting planes of the line, being perpendicular respectively to the traces of the plane, must be perpendicular to the plane itself; therefore the line, which is the intersection of these projecting planes, is perpendicular to the plane.

41. Problem. *To draw through a given point a line perpendicular to a given plane; and to find the point in which it pierces the plane.*

Each projection of the given point will of course be a point in the corresponding projection of the proposed line; and the projections of the proposed line will be perpendicular to the corresponding traces of the given plane.

Fig. 16. Let (L′, L″) be the given point (*fig.* 16), and (*M*M′, *M*M″) the given plane. Draw L′E′ perpendicular to *M*M′, and L″E″ perpendicular to *M*M″; the line (L′E′, L″E″) is the line required.

The other part of this construction refers itself to article 33.

42. Problem. *To draw through a given point, a plane perpendicular to a given straight line.*

It is manifest from the constructions which have just been performed, that the traces of the required plane will be perpendicular to the projections of the given line; if, therefore, we assume a plane which answers this condition, the problem reduces itself to the construction in article 38.

43. *Remark.* A simple method of constructing the two preceding problems, is to take the vertical plane of the projection in the proposed line (*fig.* 17), in each case. Fig. 17.

In the first problem, the plane might be given by its horizontal trace and its inclination to the horizontal plane. In the second problem, the line might be given by its horizontal projection and its inclination to the horizontal plane. Fig. 17.

The details of this construction we shall leave as an exercise for the learner.

44. PROBLEM. *To pass a plane through three points, given in space, not in the same straight line.*

Join the three points by two straight lines; find the points where these lines pierce the co-ordinate planes (18); and the straight line joining the two points in each of the co-ordinate planes will be the trace of the required plane in that plane.

Let the three given points be (M′, M″), (N′, N″), and (P′, P″), (*fig.* 18); the two lines joining these points and determining the required plane, will be (M′N′, M″N″) (M′P′, M″P″); and E′, F′, will be the points where these lines pierce the horizontal plane. The straight line E′F′ joining these two points, will consequently be the horizontal trace of the required plane. Fig. 18.

To find the vertical trace of the proposed plane, we draw, according to article 38, M′*G* parallel to E′F′, and *G*G″ perpendicular to the ground line and equal to *M*M″; the straight line *H*G″ will be the vertical trace of the required plane.

45. We may construct this problem by passing a vertical plane through one of the given points and each of the others, and revolving these two planes about their intersection with the horizontal plane, as in article 20, till they coincide with this plane (*fig.* 19). If we then draw MN meeting M′N′ in F′, and MP meeting M′P′ in E′, the straight line E′F′ will be the horizontal trace of the required plane. Fig. 19.

It is manifest that, by drawing M′G′ perpendicular to this horizontal trace, and constructing the triangle G′M′M‴ of which the angle at M′ is a right-angle and the side M′M‴ equal to M′M, the angle MG′M′ will be the inclination of the required plane to the horizontal plane (35).

46. The lines ME′, MF′, are the distances in space of the point M from the points E′ and F′ where these lines pierce the horizontal plane; we have therefore the three

Fig. 19. sides of the triangle whose vertices are M, E′, and F′. This triangle may be conceived to turn about the side E′F′ till it meets the horizontal plane on the other side of this line; its vertex M will then be at the point *m*.

47. *Remark.* In this construction we have the *developement* of the entire surface of a tetraedron whose base is M′E′F′ and whose lateral faces are ME′F′, ME′M′, and MF′M′.

48. Problem. *Two planes being given in position, to find the angle which they make with each other.*

The angle made by two planes being the plane angle made by two lines drawn in these planes through the same point perpendicular to their line of intersection; to determine this angle, we may construct a plane perpendicular to the intersection of the given planes. The intersections of this auxiliary plane with the given planes, will form an angle with each other equal to the angle of the proposed planes. This angle may be easily found from the construction in article 46.

The following very simple solution of this problem was communicated to Lacroix by Monge.

"Suppose the two given planes to be (*E*H′, *E*F″), Fig. 20. (*G*H′, *G*F″), (*fig.* 20), and the horizontal projection of their intersection to be *F*H′. We construct in the vertical plane passing through this straight line, the intersection of the two given planes, by drawing *F*F perpendicular to *F*H′ and equal to *F*F″; then through any point M′ taken at pleasure in the line H′*F*, construct a plane perpendicular to the line F H′ (43); and find also the straight line PM′, which is the intersection of this plane with the vertical plane F*F*H′. But if we revolve the first of these planes about its horizontal trace L′N′, the line PM′, being perpendicular to M′N′, will necessarily fall upon H′M′ (20), and the point P will be found in P′; the triangle whose vertices are L′, P, N′ will not be altered by this revolution; and the angle L′P′N′ will be the angle of the given planes."

49. Problem. *A plane being given, and also a line in that plane, to draw through this line a second plane making with the first a given angle.*

This problem is easily solved by retracing the several steps of the preceding solution. In this case, the data

are the plane $(EH'\ EF'')$, and the vertical plane $FH'F$ which is revolved into the horizontal plane. We construct $L'N'$ and the point $P'$, as in the preceding problem, and make upon $L'P'$ the angle $L'P'N'$ equal to the given angle; through the point $N'$ thus determined, we draw $H'G$, we then draw $GF''$, and we have the required plane $(GH', GF'')$. Fig. 20.

50. **Problem.** *Two lines which cut each other being given in space, to find their angle.*

The two projections of the point of meeting of these two lines, will be in a straight line perpendicular to the ground line (24). Construct the given lines; find the points $P'$, $Q'$, (*fig.* 21) where they pierce the horizontal plane (18); draw the straight line $P'Q'$; and this straight line together with the two given lines will form a triangle, of which the angle opposite to this side is the angle required. Fig. 21.

To measure this angle, revolve the triangle about the side $P'Q'$ into the horizontal plane. To ascertain at what distance from $M'$ the point $M$ will fall, suppose $N'$ to be the foot of a perpendicular drawn from $M$ to the base $P'Q'$ of the triangle. The line $MN'$, which measures the height of this triangle, will be the hypothenuse of a right-angled triangle whose two sides are $MM'$, $M'N'$. Revolve this triangle about the side $MM'$ till it is parallel to the vertical plane, and the vertical projection $NM''$ of its hypothenuse will be the height of the triangle $P'MQ'$. The required angle $P'MQ'$ is now easily measured.

51. **Problem.** *The projections of a line and the traces of a plane being given, to construct their angle.*

The angle which the line makes with the plane is the angle which it makes with its projection upon that plane; if, therefore, from any point in the given line, a perpendicular be drawn to the plane, it will meet this projection and a right-angled triangle will be formed; consequently the angle which this perpendicular makes with the given line, is the complement of the angle which the given line makes with the plane. This question is therefore reduced to finding the angle of the two lines.

The angle $P'MQ'$ (*fig.* 22) is the complement of the required angle, whose construction will be readily under- Fig. 22.

stood from the last article, observing that *(O*K′, *O*K″) represents the proposed plane, and that the other parts of the construction are the same as in the preceding solution.

52. Problem. *Two straight lines which do not cut each other being given in space, to find the distance between their nearest points.*

Suppose first that one of the given straight lines is per-
Fig. 23. pendicular to the horizontal plane (*fig.* 23), its projection in this plane will be the point M′ (16), and its vertical projection will be *M* M″ perpendicular to the ground line. Draw M′P′ perpendicular to the horizontal projection of the second given line, and this will be the distance sought.

53. The shortest distance between two straight lines,
Fig. 24. M′N′ and EF (*fig.* 24) may also be found, by drawing through one of them a plane H′G′ parallel to the second, and then drawing from any point in the second a perpendicular EE′ to this plane. This perpendicular is the distance sought, and determines the plane FEE′ which cuts the straight line M′N′ in the point P′, where this straight line approaches the nearest to EF.

Fig. 25. Let the line (*E*P″, E′P′) (*fig.* 25) be the first of these lines, and (*O*M, O′*M*) the second. E′ is the point where the first given line pierces the horizontal plane; and (E′L′, *E*L″) is a line drawn through E′ parallel to the second given line, to determine a plane parallel to this last. The plane passing through the line drawn as above, and through the first of the given straight lines, is constructed by a process analogous to that of article 44; this plane is G′ *G* G″; and as it is parallel to the second of the given straight lines, it is only requisite to draw, through any point of this last, a perpendicular to this plane; this is done through the point *O* O′.

We next seek the point of meeting of this perpendicular with the plane G′*G* G″ (33); this point is (N′, N″).

In order to find the nearest points of the proposed lines, we draw, through the point N′, (parallel to *M*O′ the horizontal projection of the second straight line) the line N′P′, which is evidently the horizontal projection of the intersection of the plane G′*G*G″ with a plane drawn perpendicular to this and through the second given line, since it belongs to a straight line parallel to this,

and which passes through the foot of the perpendicular drawn from this straight line to the plane in question; in other words, it is the projection of $E'F'$ in figure 24. Fig. 25.

The point $P'$ in which the line $N'P'$ meets the horizontal projection $E'P'$ of the first of the given lines, is the projection of the point $P'$ of figure 24, and consequently the projection of the point where the first straight line approaches the nearest possible to the second. The perpendicular to the plane $G'GG''$, drawn through this point, is the shortest distance sought; its projections are $P'K'$, $P''K''$, parallel to $N'O'$, $N''O$, respectively; its length is the hypothenuse of a right-angled triangle, of which one side is $P'K'$, and the other, the difference of the distances of the points $P''$ and $K''$ above the ground line. This will be found by article 21.

54. Problem. *Having two of the plane angles in any triedral angle, and the inclination of these planes, to construct the developement of this triedral angle.*

Suppose one of the given faces $Af'f$ to revolve into the plane of the other $AfE$ (*fig.* 26); if through any point $f$, Fig. 26. taken at pleasure in the edge $Af$, a perpendicular $ff'$ be drawn, it will describe in this ~~developement~~, a plane perpendicular to this edge. In this plane, the angle of inclination of the given planes will be found; if, therefore, we make upon $fF$ the angle $F''fF$ equal to the given angle, and take $fF''$ equal to $ff'$, we shall thus have the situation of the point $f'$ in relation to the line $fF$, when the planes have their given inclination. But it is evident that the three points $fF''$ and $F$, determine the base of a pyramid formed by the proposed triedral angle and the plane which has been drawn perpendicular to the edge $Af$; and the required face of this triedral angle will meet the first face in $AF$, and the second in $Af$, and will apply itself to the constructed triangle, in the line $FF''$. The third face will therefore be determined by the triangle whose three sides are $AF$, $Af'$ and $FF''$.

*Remark.* This is one case of the general problem Any three of the *six* things in a triedral angle (namely, the three plane angles and the three diedral angles) being given, to find the others. As every spherical triangle answers to a triedral angle at the centre of the sphere, (the sides of the spherical triangle measuring the plane angles in the triedral angle, and the angles of

Fig. 26. the triangle being the diedral angles in the triedral angle,) it is manifest that the solution of this general problem is the solution of all problems which can occur in *spherical trigonometry.*

As all polyedral bodies are determined by planes and their intersections, the preceding problems will enable us to construct all kinds of polyedrons.

## Section III.—*Of the Generation of Geometrical Magnitudes.*

55. If we conceive a point to move constantly in the same direction, it describes a straight line. The straight line is the path described by the moving point; that is, it consists of the *successive positions* occupied by this point. And these positions, or points, we call the *elements* of the line; and we say that the line is *generated* by the motion of the point.

It will be readily seen that the number of these elementary positions in a finite line, is illimitable; we therefore say, that the line is composed of an infinite number of points.

If a straight line be conceived to move in such a manner that any two points in it shall describe equal and parallel straight lines, it will be always in the same plane, and will generate a parallelogram. This surface will be composed of the successive positions of this straight line; and these positions or *lines* constitute the *elements of the surface.* And all plane surfaces may be considered as composed of *right-line* or *rectilinear elements.*

A circle may be considered as composed of an infinite number of *parallel* straight lines. It must then be considered to have been generated by a straight line (its diameter) moving in a plane and changing its magnitude with its distance each way from its first position, according to a certain law.

Or, as a circle is generated by the motion, in a plane, of a straight line equal to radius, about one extremity (El. 266); each elementary point in this moveable radius may be considered as generating a circular curve. These circular curves compose the entire surface of the circle, and constitute its *curvilinear* or *circular* elements.

56. The motion of a surface, out of the plane of its original position, generates a body.

If a *square* move in such a manner that any three points in it, not in the same straight line, shall describe parallel straight lines, each equal to a side of the square and perpendicular to the surface, the product of this motion will be a cube; and the successive positions of this square are the elements of the cube.

If the several points in the square should move in straight lines passing through the same fixed point, the dimensions of the square will be in a constant ratio to its distance from that point; and must therefore all be zero (0) at the instant of passing that point. The body thus generated by the moving square will be a pyramid whose summit is the fixed point, whose base is the square, and whose elements are the successive positions and magnitudes of this square.

A triangular or any other polygonal pyramid may be conceived to be generated in the same manner. The analogous method of generating the cone and cylinder, was noticed in the Elements (El. 266, 275).

A sphere may also be considered as generated by the *direct* motion of its great circle always parallel to itself; its centre describing a straight line perpendicular to its plane; and its area diminishing with its distance (each way) from its original position, according to a certain law.

In these examples, every successive position of the generating plane is a section of the body thus generated. And as these successive positions constitute the whole intermediate space, and are all parallel, the body may be considered as composed of an infinite number of parallel sections.

*Of Curves.*

57. If a point in motion continually change its direction, its path is a *curve* line. If this line lie wholly in the same plane, it is called *a curve of single curvature.*

If the points of the curve are not in the same plane, it is called *a curve of double curvature.*

58. Mathematical curves are considered to be given, when the law, according to which the generating point changes its direction, is given.

A point in motion will generate a *circular curve,* when

the conditions of its motion are that it shall continue in the same plane, and at a constant distance from a given point in that plane.

If the conditions of its motion are, that it shall continue in the same plane, and that the *sum* of its distan-
Fig. 27. ces MF, M*f*, from two fixed points F, *f*, (*fig.* 27) is constantly the same, it will generate a curve called an *ellipse.* The longest diameter of the ellipse is called its *transverse axis;* the shortest is called its *conjugate* axis. These axes are perpendicular to each other; and each divides the curve into two parts which are equal, and symmetrical with respect to this axis. The points of the ellipse at the extremities of the axes, are called the *vertices* of the ellipse.

If the condition is, that, continuing in the same plane,
Fig. 28. the *difference* of the two distances MF, M*f* (*fig.* 28) of the generating point from two fixed points shall not vary, it will describe a curve called *a hyperbola.* The *axis* of the hyperbola is a straight line dividing the curve into two symmetrical portions. The point where the axis meets the curve, is called the *vertex* of the hyperbola.

If the point move with the conditions, that it shall not depart from the plane, and that its distance MX from a
Fig. 29. straight line, as XZ (*fig.* 29) drawn in that plane, and its distance MF, from a fixed point F in that plane, shall be constantly equal, the curve generated by the point will be a *parabola.* The axis of the parabola is a straight line dividing the curve into two symmetrical parts, as in the hyberbola; the point where the axis meets the curve is called the vertex of the parabola.

These four curves, the *circle,* the *ellipse,* the *hyperbola,* and the *parabola,* may be obtained by cutting a right cone with a plane making different angles with the axis of the cone.

(1). If a right cone whose base is a circle, be cut by a plane perpendicular to its axis, the section will be a circle (El. 268).

(2). If the cutting plane make an acute angle with the axis, *greater* than that which the side of the cone makes with the axis, the section will be an ellipse.

(3). If the cutting plane make with the axis of the cone, an angle *less* than that which one side makes with the axis, and do not pass through the vertex, the section will be a *hyperbola.* In this case the cutting plane must be parallel to two right-line elements of the conical surface.

These elements, therefore, would not be cut by this plane, though all the other elements would. And if these parallel elements were projected upon the cutting plane, their projections could not touch the curve, though they would continually approach it, as we follow the curve down from the vertex. These projections are called *asymtotes* of the curve.

It is important to observe that the plane, in the case of the hyperbola, though it cuts all the right-line elements of the surface excepting two, it does not cut them on the same side of the *apex* of the cone. But these elements being produced, the plane will cut the remainder *beyond* the apex. The curve, therefore, consists of two *branches:* The straight line joining their vertices is called the *transverse axis;* and a straight line perpendicular to the middle of this, and in the plane of the curve, is generally assumed as the *conjugate axis* of the hyperbola.

(4). If the angle which the cutting plane makes with the axis of the cone, is *equal* to that which the side makes with the axis, the section is a *parabola.* The parabola has but one axis.

These curves, on account of this method of obtaining them, are usually called the *Conic Sections.*

59. If a point be conceived to move in the surface of a cylinder or a cone, with this condition, that its path shall, in every part, make the same angle with the axis of the cone or cylinder, it will, in each case, describe a *spiral curve,* which is a curve of *double curvature.* The more important curves of double curvature are those produced by the intersection of curve surfaces.

## *Of Curve Surfaces.*

60. The curve surfaces of which geometry takes cognizance, are generated by lines, according to certain mathematical laws which serve to determine their parts and the character of their curvature.

As a plane surface is considered as consisting of an infinite number of straight lines, so we may conceive a curve surface to be composed of an infinite number of curves; and this surface will be perfectly determined, when we have these curves and the law which connects

each one of them with the next. A curve line, therefore which moves in space, or which changes, at the same time, its magnitude and position, according to some determinate law, generates a curve surface. This surface may be considered as composed of the successive positions and forms which the line takes during its motion.

61. The most simple of curve surfaces are those of the cone and the cylinder.

A *cylindrical surface* is generated by a straight line moving in space, always, parallel to itself, and directed in its motion by a curve line. The straight line is here called the *generatrix,* and the curve line the *directrix.* If the directrix be a circular curve, the surface will be that of the cylinder considered in the *Elements.*

The cylindrical surface may also be generated by the motion of a curve line along a straight line, each point in the curve moving in a direction parallel to the straight line. In this case the curve line is called the *generatrix*, and the straight line the *directrix.*

If the cylindrical surface be cut by a plane parallel to the right-line generatrix, the section will be a straight line coincident with one position of this generatrix. The successive positions of this right-line generatrix we call the *right-line elements* of the cylindrical surface.

If this surface be cut by parallel planes making any angle with the right-line elements, these sections will be all the *same curve;* and each of these is an element of the cylindrical surface.

62. If the motion of the right-line generatrix were determined by the conditions, that this line should pass constantly through the same point, and that it should move along a curve line, the product of this motion would be a *conical surface.* If the curve line were the circumference of a circle, the conical surface thus produced would be that of the cone considered in the *Elements.* But it is not essential to a cylindrical or conical surface, as the terms are used in Descriptive Geometry, that the curvilinear directrix should be of a nature to return into itself.

Cones and cylinders are distinguished by their bases; they are called *circular, elliptical, hyperbolical,* or *para-*

*bolical*, according as the base is a *circle*, an *ellipse*, a *hyperbola* or a *parabola*.

The conical surface may also be conceived to be generated by the motion of the curve which was before taken for the directrix; the conditions being, that all the points in the curve shall move in straight lines converging to a fixed point: this point will be the summit of the cone.

63. As the right-line generatrix is considered as infinite, it is manifest that the conical surface generated by its motion will consist of two parts perfectly symetrical, but in opposite directions from their common summit. These two parts (called, in French, *nappes)* are considered as constituting but one and the same surface.

If a conical surface be cut by a plane passing through the summit, the section will be in a right-line element of the surface; if it be cut by planes parallel to each other, none of which pass through the summit, these several sections will be similar curves, and are considered as curvilinear elements of the surface.

64. If the straight-line generatrix, instead of moving according to the conditions given for the cylinder or the cone, were to move along two straight lines or a straight line and a curve line not in the same plane, or along two curves not in the same plane and so situated that the straight line shall neither pass constantly through the same point, nor have its successive positions parallel, the surface generated by its motion will be what is called a *warped surface.*

65. It will be at once seen that there may be an infinite variety in the character and curvature of surfaces which fall under the general division of warped surfaces. We shall give one or two examples.

(1.) Let the right-line generatrix have two right-line *directrices;* suppose that one of them is perpendicular to a horizontal plane, and the other inclined to this plane; and suppose that a condition of the motion of the generatrix, is, that it shall be constantly parallel to the horizontal plane. It will be readily perceived what is the kind of surface generated by this motion of the straight line.

(2.) Let there be one right-line directrix in the vertical plane, and perpendicular to the horizontal plane; and let the other directrix be a circular arc of 90°, whose plane is perpendicular to each of the co-ordinate planes, whose centre is in their common intersection, and whose extremities are in these planes respectively; and suppose the right-line generatrix to be always parallel to the horizontal plane. The body embraced by the surface generated by this straight line, and the two co-ordinate planes, is what was called by Wallis, a *conoidal wedge*.

66. Warped surfaces, although generated by straight lines, differ from cylindrical and conical surfaces, in a very important particular. The latter are capable of being *developed* or *unrolled* upon a plane; the former are not capable of such *developement*.

That any surface may be developed, it is necessary that any two contiguous positions of the right-line generatrix should be in the same plane.

67. To understand the developement of a cylindrical surface, suppose a plane applied to it; it is manifest that the plane will touch the cylindrical surface along one of its right-line elements. Let the cylinder be *rolled* upon the plane, the right line elements of its surface will come successively in contact with the plane, and when the cylinder has rolled once round, every right line element has touched the plane; and the portion of the plane thus brought into contact with the cylindrical surface, in one entire revolution, is equivalent to the cylindrical surface, and may be considered its *developement*. Or suppose a piece of plane paper rolled about a cylinder, so as just to cover it, to be unrolled into a plane; it will represent the developement of the cylindrical surface.

The developement of a conical surface may be illustrated in a similar manner.

68. A more common method of illustrating the developement of these surfaces, is to begin with the prism and pyramid. One of the lateral faces of the prism, for instance, is applied to the plane upon which the surface is to be developed. The body is then made to revolve about the *edge* common to this face and the next, till this second face comes into the same plane; the body is then

revolved about the edge which separates the second face from the third, till this third face comes into the same plane. The process is continued, the body revolving upon each successive edge, as upon a *hinge*, till each of the faces has been in contact with the plane. The portion of the plane thus brought in contact with the prismatic or pyramidal surface, constitutes the developement of this surface.

To apply this process to cylindrical and conical surfaces, we have only to consider the cylinder as a prism, and the cone as a pyramid, each of an infinite number of lateral faces.

69. If, instead of the right-line generatrix employed in the production of cylindrical and conical surfaces, we employ a curve line, the plane of which, in the progress of its motion, always cuts the plane of the curvilinear directrix in a line perpendicular to the tangent to the directrix at the point of intersection of the generatrix and directrix, the surface generated will be a *surface of double curvature.*

70. The most simple of *double curve surfaces*, are those which are also *surfaces of revolution.*

Of surfaces of revolution we have, in *The Elements*, three examples: those of the right cone, the right cylinder, and the sphere. Of these the surfaces of the cone and the cylinder, are single-curve surfaces. The spherical surface is both a double curve surface and a surface of revolution.

71. If, instead of the semicircle used in the production of the sphere, we employ a *semi-ellipse* revolving about one of its axes, the surface thus generated will be a *spheroidal* surface; the body thus produced is usually called a *spheroid*, sometimes an *ellipsoid.* If the semi-ellipse revolve about its *transverse axis*, the product of its revolution is called a *prolate spheroid;* if the semi-ellipse revolve about its *conjugate axis*, it will generate an *oblate spheroid.*

72. If the curve-line generatrix be a hyperbola revolving about either axis of the curve as an axis of revolution, the surface generated by this motion will be a *hyperboloidal surface;* the body embraced by this surface begin a *hyperboloid.*

73. If the generatrix be a parabola revolving round the axis of the curve, the surface generated will be a *paraboloidal* surface, the body thus generated being a *paraboloid.*

These three last bodies have been called *conoids*, on account of their being generated by the conic sections.

These conoidal surfaces are surfaces of double curvature as well as surfaces of revolution.

74. There is manifestly an infinite variety of surfaces of revolution: But they all agree in one particular; namely, that every plane section perpendicular to the axis of rotation, gives a circular curve.

A plane section of these surfaces through the axis, will always give the generating curve. This section is called the *meridian section* of the surface; or its *meridian curve*.

75. Another class of surfaces are called *annular surfaces;* these admit of great variety; the simplest is that of the *common ring*, which will serve to give a general idea of the class. This surface is generated by the motion of one circle along the circumference of a greater circle; the centre of the generating circle moving in the circumference of the directrix, with its plane at each point perpendicular to the tangent of this directing circumference.

In surfaces of this class, it is not necessary that either the generatrix or directrix should be circular, or that the directrix should be a curve of single curvature. We
Fig. 30. have, in figure 30, another example of this kind of surfaces; GH is the generating curve, and XZ is the curvilinear directrix.

## Section IV.—*Of the Projection of Curve Lines and Curve Surfaces.*

76. It is evident that plane curves parallel to the plane of projection, will have, for their projection, curves *equal* to themselves: That is, the horizontal projection of a circle, parallel to the horizontal plane, is an equal circle: The horizontal projection of an ellipse parallel to the horizontal plane, is also an equal ellipse; and so of any other curve.

But if plane curves are parallel to one of the co-ordinate planes they must be perpendicular to the other;

their projections, therefore, upon this other, will be straight lines.

77. To project, upon the horizontal plane, a plane curve parallel to it, perpendiculars are drawn to the plane from every point in the curve. These perpendiculars will form a cylindrical surface of which the proposed curve is a perpendicular section. And, in general, any curve line not perpendicular to the plane of projection, is projected by a cylindrical surface passing through the curve perpendicular to the plane of projection. This cylindrical surface is called the *projecting cylinder* of the curve. The projection is the intersection of this cylindrical surface with the plane of projection.

78. If the plane curve to be projected be inclined to the plane of projection, the projection will be a curve different from the proposed curve.

To illustrate this principle, let us suppose a circle perpendicular to the vertical plane and inclined to the horizontal. One diameter of the circle will be perpendicular to the vertical plane and therefore parallel to the horizontal plane. The horizontal projection of this diameter will be a straight line equal to itself (22); and all the chords parallel to this diameter, will have their projected lines equal to themselves. But the diameter, perpendicular to that which is parallel to the plane of projection, will have the same inclination to this plane that the circle itself has to this plane; it will therefore have for its projection a line shorter than itself, whose magnitude depends upon the degree of inclination (22). So also the chords parallel to this diameter, will have for their projections straight lines bearing the same ratio to the chords, themselves as the projection of this diameter bears to the diameter itself. That is, the right-line elements of the circle taken in the direction of their greatest inclination to the plane of projection, have for their projections lines which bear a constant ratio to these elements.

Now it is shown, in treatises on Conic Sections, that the straight lines, parallel to the conjugate axis, in an ellipse, have all the same ratio to the corresponding chords of the circumscribed circle, that is, the ratio of the conjugate to the transverse axis. *The projection of a circle*

*inclined to the plane of projection, is therefore an ellipse whose transverse axis is equal to the diameter of the circle, and whose conjugate axis is the projection of that diameter which has the greatest inclination to the plane of projection.*

This is sufficient to illustrate the principle, which the reader may apply to the ellipse and to other plane curves.

79. Curves of double curvature may also be determined by their projections upon two planes, as every point in the curve may be determined by its two projections. But those which result from the intersection of curve surfaces, will be determined by the projections of the intersecting surfaces.

80. Curves of double curvature have their parts frequently determined by the rectilinear tangents to points in these parts. The relation of a right-line tangent, to a curve of double curvature, is not easily explained. The rectilinear tangent to a plane curve, must not cut the curve and *must be in the plane of the curve.* Without this last condition an infinite number of straight lines might be drawn touching a plane curve in any point, conformable with the first condition: So also an infinite number of straight lines may be drawn through any point of a curve of double-curvature, without cutting the curve; yet only one of these can be a *tangent* to the curve. To determine the position of a rectilinear tangent to curves of this kind, the tangent has been defined to be "*a straight line passing through two contiguous points in the curve.*" It is perhaps as philosophical to fix the position of the tangent by saying that, *it must be in the plane determined by the point of contact and the two contiguous points of the curve.*

81. A cylindrical surface is determined when any plane section of the surface and any right line element are determined by their projections; as the whole surface is composed of right-lines of each of which we have one point (namely, that in which it is cut by the plane making the given section) and the direction which is parallel to the given element. Any point in the surface may therefore be determined by article 23.

A conical surface is determined by the projection of its summit and any curvilinear section, upon each of the co-ordinate planes.

82. A surface of revolution is most easily constructed by choosing one of the planes of projection perpendicular, and the other parallel, to the axis of revolution. Its projection upon the plane to which its axis is parallel, will be a meridian section. This, together with the projection of this axis upon the other plane, will enable us to determine every point in the surface.

### *Of Tangent Planes to Curve Surfaces.*

83. The theory of tangent planes to curve surfaces is important, as it enables us to determine the relative positions of these surfaces, when it could not be done by the surfaces themselves.

A plane is tangent to a curve surface when it touches the surface, and lies wholly on the same side of the surface.

Suppose a convex surface to be cut by planes all passing through the same point in that surface; the sections would be plane curves and the rectilinear tangents to these curves at their common point, would be all in the same plane. Any two of these right lines determine this plane; and this plane is tangent to the surface at this point. So that the construction of a tangent plane to any curve surface, reduces itself to the construction of two tangents to the curves in the surface at that point.

84. Suppose a straight line perpendicular to the tangent plane at the point of contact; this line is called *a normal* to the curve surface at that point: as also the normal to a plane curve at any point, is a line in the plane of the curve, drawn through the point of contact and perpendicular to the tangent.

If through the normal to a curve surface, at any point planes be supposed to pass, cutting the surface, these planes will be *normal to the surface at this point;* they are called *normal planes.*

85. Problem. *To draw a tangent plane to a cylinder.*

A plane which touches a cylindrical surface, must, from

the nature of the surface, meet it in one of its right-line elements; but it will meet the plane of the base in a
Fig. 31. straight line, as PT (*fig.* 31,) which will be a tangent to the periphery of the base.

If therefore we draw through the proposed point M, the straight line MP parallel to the generatrix AD, and construct the tangent PT to the point P of the curvilinear directrix, the tangent plane will be determined by the two lines PM and PT.

*Remark* 1*st*. If the proposed point through which the tangent plane is to be drawn, is *without* the cylinder, it is evident that there will be two planes which will answer to the conditions of the problem. To construct this solution find the point where a straight line passing through the given point and parallel to a rectilinear element of the cylinder, meets the plane of projection. A line drawn through this point, tangent to the intersection of the cylindrical surface with this plane, will be the trace of the tangent plane upon this plane of projection.

*Remark* 2*d*. If we would find the rectilinear tangent to a curve made by the intersection of two curve surfaces, we should construct tangent planes to these surfaces at the proposed point; and the intersection of the two planes would be the right-line tangent required.

86. Problem. *To construct a tangent plane to a cone.*

The only difference between this construction and that of the last problem, is, that the line MP must be drawn through the summit S instead of being parallel to
Fig. 32. the generatrix (*fig.* 32).

87. Problem. *To draw a tangent plane to a surface of revolution.*

The most convenient method of determining a tangent plane to surfaces of this class, is by means of the rectilinear tangents to the meridian curve and circular section, at the proposed point of contact.

Fig. 33. The plane tangent to the point M (*fig.* 33) will therefore be determined by the straight lines MT and M*t*; the first being tangent to the generating curve MX, and the other to the circle MZ. If therefore we can draw a tangent to the meridian curve at the proposed point of contact, it is easy to construct the plane tangent to the surface at that point.

*The tangent plane to a surface of revolution, is always perpendicular to the meridian plane passing through the point of contact.* It therefore follows that, *a meridian plane to a surface of revolution is always normal to that surface.*

### *Plane Sections of Curve Surfaces.*

88. Plane sections of a cylindrical surface parallel to its axis and of a conical surface through its summit, are all straight lines (61, 63).

Plane sections of these surfaces parallel to the bases of these bodies are similar to these bases (El. 268, 275).

In the right cylinder and right cone these sections are perpendicular to the axis ; if the bases are circles these sections will be circles. But an oblique section of a right cone by a plane which cuts all its rectilinear elements on the same side of the apex, is an ellipse (58). And it is easy to show to those acquainted with the properties of this curve, that a plane section of a right cylinder, oblique to its axis, is also an ellipse.

These sections are easily determined; for having these surfaces given, their rectilinear elements are known ; and knowing these right lines and the cutting plane, the curves themselves are determined by finding the points where these lines meet this cutting plane, (33).

89. A plane section of a sphere is always a circle. The determination of these sections belongs to the solution of the general problem—*To find the position and magnitude of the circle which is the intersection of a given sphere and a given plane.*

By drawing a perpendicular from the centre of the sphere to the cutting plane, and determining the meeting of this line and the proposed plane, we have the centre of the circle required.

This solution will be very simple, if we take the plane of the vertical projections, DAB (*fig.* 34) perpendicular **Fig. 34.**
to the horizontal trace of the proposed plane, which may always be done. Then, O′ and O″ being the projections of the centre of the sphere, if we suppose it cut by a vertical plane drawn through the line H′ O′ perpendicular to AC; this plane will pass through the centre of the

Fig. 34. sphere, and will contain the perpendicular drawn from this point to the proposed plane DAC; but it is parallel to the vertical plane DAB: We may therefore suppose it to be applied to this vertical plane without any change of the lines which it contains, either in magnitude or in position relative to its horizontal trace $H'O'$ which will then coincide with AB. This being premised, $M''E''N''$ will be the great circle which results from the section of the sphere by the vertical plane just mentioned; the perpendicular $O''G''$ will determine the vertical projection $G''$ of the centre of the section sought. We thence obtain its horizontal projection $G'$; and, revolving the plane DAC about it horizontal trace, as this centre will fall upon the point G in the horizontal plane, we may describe from this point, with the radius $G''M''$, the circle MN which will be the required intersection of the sphere and plane proposed.

## *Of the Intersections of Curve Surfaces.*

90. The most simple method of constructing the intersections of curve surfaces, is to suppose a series of plane sections of the two surfaces, and to determine the section made by each plane with each of these surfaces; the points common to the two curves, in any one of these plane sections, are of course points in the intersection required.

This method will be the most simple if we draw the cutting planes parallel to one of the co-ordinate planes. Suppose them parallel to the vertical plane; the sections which they make with the proposed surfaces will have their horizontal projections respectively in the horizontal traces of these cutting planes; and their vertical projections will not only show the height of any required point above the horizontal plane, but will be exactly equal to the auxiliary curves whose respective projections they are.

A few examples will illustrate these principles, and enable the reader to apply them generally to the construction of analogous problems.

91. Problem. *To construct the intersection of a cylinder and a sphere.*

We shall take for the plane of the horizontal projec-

tions a plane perpendicular to the rectilinear elements of the cylinder; the vertical plane will therefore be parallel to these elements. We suppose cutting planes parallel to the vertical plane; their intersections with the cylindrical surface will be right-line elements of this surface; and their intersections with the sphere will be circles whose centres will have their vertical projection in the same point with that of the centre of the sphere. The radii of these circles may be easily found from the discussion in article 89.

The details of these constructions will be mostly left to the reader, as they will be found easy when the principles already discussed are well understood.

In the present case we draw straight lines $G'I'$, $g'i'$, (*fig.* 35) in the horizontal plane, parallel to the ground line. Fig. 35. These lines are the horizontal traces of the auxiliary planes, which are supposed parallel to the vertical plane of projection; $g'i'$ is the radius of the circle which is the section made by one of these planes in the sphere whose centre has its horizontal projection in $E'$ and its vertical projection in $E''$. The points $P'_1$, $P'_3$, where the line $g'i'$ meets the base of the cylinder, are the horizontal projections of two rectilinear elements of this surface which are met each in two points by the circumference of the circle which is the section of the sphere by this auxiliary plane whose horizontal trace is $g'i'$. If, therefore, from the point $E''$ as a centre and with a radius equal to $g'i'$, we describe a circumference, the points $p''_1, p''_3, P''_1, P''_3$, where it meets the vertical projections of the rectilinear elements just mentioned, will belong to the intersection of the cylinder and sphere.

By a similar process we may determine as many other points as we wish of this intersection. There is, however, one circumstance in the general problem under discussion, which demands particular attention; this is the case in which the cylinder *entirely penetrates* the sphere, which it will do if the radius of a circular section of the cylinder is less than the radius of the sphere minus the distance of the axis of the cylinder from the centre of the sphere. In this case there will be two intersections of the surfaces in question, the one where the cylinder enters the sphere, and the other where it emerges from it.

The horizontal projections of these two curves will of course be the same; and it will be readily seen that the

Fig. 35. parts $P''_1$, $P''_3$, $P''_4$, $P''_2$, belong to the vertical projection of the curve, where the cylinder enters the sphere; and that $p''_1$, $p''_3$, $p''_4$, $p''_2$, belong to the vertical projection of the curve where the cylinder emerges from the spherical surface.

Fig. 36. The following figure represents the case in which the distance of the axis of the cylinder from the centre of the sphere is greater than the difference of the radii of the sphere and cylinder. In this case the cylinder does not entirely penetrate the sphere; there is therefore but one intersection of the two surfaces.

92. Problem. *To find the projection of the curve made by the intersection of the sphere and the cone.*

Suppose the cone to be cut by planes passing through its apex, and perpendicular to the horizontal plane. The sections thus made in the conical surface, will be straight lines which are easily determined; the sections which these auxiliary planes make in the sphere, will be circles whose centres and radii are also easily found (89).

Fig. 37. Let S′K′ (*fig.* 37) be the horizontal trace of one of these cutting planes. Instead of revolving about this trace to coincide with the horizontal plane, we suppose it applied to the vertical plane DAB by bringing the line S′K′ to AB, so that the point S′ will fall upon *S*; then by taking *Sk* equal to S′K′ we can draw the straight lines S″*k*, which will be the sections of the conical surface by the cutting plane. Making *mg* equal to S′G′, the point *g* will be the position of the centre of the circle in which the sphere is met by the cutting plane, and whose radius is *gh* equal to G′H′, (89).

The points *p* where the circumference of this circle meets the straight lines S″*k*, belong to the intersection of the proposed cone and sphere. There are four of these points; two where the cone enters the sphere, and two where it emerges from it.

The points *p* are situated in the cutting plane. To have their projections, we must take S′P′ equal to *pi*, and the point P′ will be the horizontal projection; by drawing P′P″ perpendicular to AB, the vertical projection P″ will be found at the meeting of this straight line with *pi*; for the point *p* being taken in a vertical plane, is at the same height with its projection upon every other vertical plane.

That we might not render the figure two complicated we have performed this process upon only one of these points P; it is readily seen, however, that the process is applicable to the other three. Fig. 37.

93. Problem. *To construct the intersection of two cones.*

Suppose that the proposed cones have their bases in the same plane, or (which is the same thing) that we know the curvilinear section made in each of these cones by one of the co-ordinate planes, the horizontal, for example. The problem may always be reduced to this state.

Now imagine a plane passing through the line which joins the summits of the proposed cones, and turning about this line; this plane in each of the positions where it meets the cones will cut each of them in two rectilinear elements of their surfaces; and as these are all in one plane, the two which belong to the first cone, will meet those which belong to the second; and these points of meeting will be points in the intersection required.

Let $(S', S'')$, $(s', s'')$, (*fig.* 38) be the summits of the cones, $F'F'$ and $f'f'$ the curves which are their intersections with the horizontal plane, E the point where the line joining their summits meets the horizontal plane; it is evident that the cutting plane, in all its positions, will pass through this point. Fig. 38.

We now draw the straight line $E'F'$ at pleasure, but in such a manner that it will meet the bases of the cones, and consider this line as the horizontal trace of the cutting plane.

We next construct the projections of lines drawn from the points $F'$ to the summit of the first cone, and from the points $f'$ to the summit of the second; these lines are respectively the projections of rectilinear elements of the conical surfaces proposed, situated in a plane which passes through the straight line which joins the summits of the cones, and through the straight line $F'F'$; their points of meeting marked upon each of the co-ordinate planes by the figures 1, 2, 3, 4, will be points in the intersection required.

The figure represents a case in which one of the cones entirely *penetrates* the other. Two of the four points found by the preceding construction, belong to that part

of the curve where the *penetrating* cone enters the other, and the other two points belong to the part where the first cone emerges from the second.

94. To find the intersection of a cone and cylinder, we imagine a straight line passing through the summit of the cone parallel to the axis of the cylinder; then planes passing through this line will cut the proposed conical and cylindrical surfaces in their recti-linear elements.

95. To find the intersection of two cylinders, it will be necessary to cut these bodies by planes parallel to their axes; if the cylinders have their bases upon the same plane, the construction will be analogous to that given above for the cones.

We first determine the horizontal traces of the cutting planes, which we do by passing through any point two straight lines parallel respectively to the axes of the two cylinders; the horizontal trace of the plane of these lines will of course be parallel to the horizontal traces of the cutting planes. We may draw as many of these traces as we wish; the points where they meet the periphery of the base, will be points in the rectilinear elements according to which these surfaces are supposed to be cut by these auxiliary planes; these right lines being constructed will give, by their mutual intersections, points in the required intersection.

## Section V. *Linear Perspective.*

96. Perspective is a science which teaches us to represent upon any surface whatever, the outline of objects, such as they appear when viewed from any given point.

Light in passing through a homogeneous medium moves in straight lines; and objects become visible by means of the rays of light which proceed from their surfaces to the eye. These rays, by their inclinations among themselves, determine the *images* of bodies.

Thus we perceive the *contour*, or apparent outline, of
Fig. 39. the quadrilateral ABCD (*fig.* 39) because from each point of it a ray of light is conveyed to the eye. It is manifest that these rays, taken together, constitute the

pyramid formed by the lines drawn from the different points of the object to the eye.* Fig. 39.

Let O-ABCD represent this pyramid of rays, the summit O being the place of the eye. Each of these rays or rectilinear elements of this pyramid, must appear to the eye but *a single point.* If, therefore, this pyramid be cut by a plane or by any other surface, this section will exhibit to the eye at O the same outline as the quadrilateral ABCD. It is not necessary, therefore, in order to give us the sensation produced in us by the organ of vision, that the object itself should be presented to the eye; it is sufficient for this purpose, to determine an assemblage of rays disposed in the same manner respectively as those which pass to the eye from the different points in the object.†

Hence we can represent objects upon a plane; for if we conceive the pyramid formed by the assemblage of rays transmitted from different parts of the object to our eye, to be cut by a plane, an image would be formed which would represent the contour of the body and the relative position of its differents parts.

It follows from what precedes that the determination of this image depends entirely upon finding the intersections of the lines proceeding from the eye to different conspicuous points of the object, with the plane or surface on which it is to be represented.

This surface is called the *picture* or *plane of delineation.* The respective positions of the eye, the picture, and the object, must be determined, in order that the image may be determined.

The knowledge of the true form and dimensions of

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* This supposes the object to be either white or colored, but not black; for in that case it would be perceived only by the absence of light; thus we might say that the pyramid was determined by the absence of rays from the space occupied by the quadrilateral.

† It is evident that a perspective of the object would also be formed, supposing the visual rays produced beyond it and extended until they meet the plane situated behind it; the image in this case would be greater than the object.

the body which we wish to represent, will give us the projections of the conspicuous points which determine its contour, and the situation of the parts which compose it.

The problem will then be reduced to finding upon the plane of delineation, the image of each of these points, or in other words, the meeting of a given straight line with a given plane.

We shall discuss some of the different cases which the problem presents.

97. PROBLEM. *To find upon the plane of the picture, situated in any manner whatever, the appearance or the perspective of a point given in space.*

Take the vertical projection of the proposed point, on a plane perpendicular to the common intersection of the plane of the picture with the horizontal plane. Let
Fig. 40 T′AT″ (*fig.* 40) be the plane of the picture; O′ and O″ the projections of the eye O; P′ and P″ those of the point P which is to be put in perspective; O′P′ and O″P″ will be the projections of the visual ray OP.

The meeting p of this line with the plane of the picture, will determine the perspective sought, which may be found by article 33; but as this point must be constructed upon the plane of the picture, the projections p′ and p″ are not sufficient. This meeting of the visual ray with the plane of delineation is called *the perspective* of the point from which the ray emanates, and is really an *oblique projection* of this point upon the plane of the picture. As the lines by which the several points in the object should be projected upon this plane, must converge to the point in which the eye is situated, this projection is called a *perspective projection.*

We therefore draw p′*p* through the horizontal projection of the required point *p*, and perpendicular to the horizontal trace of the plane of the picture. We now have the distances A*p* and Ap″ of the required point from the two lines AT″ and AT′ perpendicular to each other in the plane of the picture.

The line AT′, which is ths intersection of the plane of the picture with the horizontal plane, is called in perspective the *base line;* it is considered as limiting the bottom of the picture, and as exhibiting the ground on which the original object stands.

98. When the plane of delineation is perpendicnlar to the horizontal planc, as the plane T′A t″, then the projections O′P′ and O″P″ themselves determine, by their intersections with the lines TA and t″A, the distances A q′ and A q″ of the perspective q from each of these straight lines. Fig. 40.

We take for example a pyramid (*fig.* 41) of which the four triedral angles have their summits projected at the extremity of the rays drawn from the points O′ and O″. The construction of the perspective of one of these summits is designated by the same letters as in figure 40. Fig. 41.

In the case where the plane of delineation is vertical, the construction is very much simplified by taking the plane of the picture itself for the co-ordinate vertical plane. The eye, being supposed behind the plane of the picture, (*fig.* 42) has its horizontal projection in O′; that of the point in question is in P′, and p is the perspective of this point. Fig. 42.

99. *Remark.* If the object to be represented is terminated by straight lines and planes, we may construct its image by seeking the perspectives of the vertices of the polyedral angles by which it is terminated; and in order to this, it will only be necessary to repeat the process which has just been indicated. Two points will determine a straight line, and the faces of the proposed object will be formed by a certain number of lines.

When the object is terminated by curve surfaces, no particular point is presented by which we may determine its form; we must first find its *visible limit.* The visible limit of a body is the curve which separates the part which is seén from that which is not seen; it is evidently formed by the series of points in each of which the visual ray merely touches the surface of the body. If we conceive a conical surface having its summit placed at the eye, enveloping the proposed body, by touching it, the curve of contacts will be precisely that of the visible limit. If we cut this cone by planes drawn through the eye, in any manner whatever, each of them will form in the proposed body a section to which two of the right line elements of the cone will be tangent. From this results a general method of constructing the visible limit of a curve surface.

Let us suppose this surface to be cut by a series of ver-

Fig. 40. tical planes such as OO'P'P, (*fig.* 40) passing through the eye; construct upon the vertical plane the projection P''X'' of each of the sections, and from the point O'' draw O''P'' tangent to this curve. Having the projections of the visual ray, we can find as in the preceding problem, the perspective of the point P situated upon the visible limit of the proposed object.

100. If we suppose the eye to be situated at an infinite distance from the object, so that the visual rays may be considered as parallel; having designated by a straight line the direction in which the body must be seen, nothing more remains to place the points in perspective but to draw from these points, lines parallel to the given line, and to find their intersection with the plane of delineation.

It is readily seen that in this hypothesis, the apparent contour of the body is determined by tangents to its surface which are parallel to each other, the whole of which, taken together, form a cylindrical surface. To determine these tangents we choose cutting planes vertical and parallel to the given line; and the tangents to the vertical projections of the sections must be drawn parallel to the given line which marks the direction of the visual ray.

This perspective has a great analogy to the orthographic projection discussed in the former part of this introductory treatise; and may be used in constructing problems of the same kind with those which were solved in the second section. There is no necessity for making these projections by perpendicular lines; in many instances the solutions would be as simple if the projections were made by oblique lines.

I shall proceed to give some propositions which may serve as the foundation of another method of perspective which may be applied with great facility to bodies terminated by planes and straight lines.

101. *If we draw through the eye a straight line parallel to a straight line situated in any manner whatever with respect to the plane of delineation, the point where this parallel meets this plane belongs to the perspective of the straight line proposed.*

In fact, all the lines drawn from the eye to the dif-

ferent points of any proposed straight line, form a plane which, by its intersection with the plane of the picture, determines the perspective of this straight line; but the line OO′ (*fig.* 43) being parallel to the proposed line and passing through the eye which we suppose in O, is situated necessarily in this plane; therefore, the point O′ where it meets the plane of delineation TA, belongs to the perspective of the straight line proposed. Fig. 43.

It is evident that the point P′ where the proposed straight line meets the picture itself, makes also a part of its own perspective; therefore, to trace this perspective, it is sufficient to know the points where the proposed line and a line drawn parallel to it through the eye, will meet the plane of the picture.

102. It follows from the preceding article, that the perspectives of any number of lines parallel to each other, will all meet in a single point in the plane of the picture; this point is called in treatises on perspective, their *vanishing point.* Only one line, indeed, can be drawn through the eye, which will be parallel to all these others; their perspectives must therefore pass through the point where this line meets the plane of the picture.

103. If the proposed lines were at the same time parallel to the plane of the picture, the straight line OO′ drawn through the eye, would not meet the picture; and consequently their perspectives would be parallel among themselves. We may be convinced, *a priori*, of the truth of this proposition, by the following reasoning: The two proposed straight lines being parallel, the planes formed by the assemblage of rays drawn from the eye to different points of these lines, and containing their perspectives, have necessarily their intersection parallel to these same lines, (El. 214) and consequently, to the plane of the picture. The perspectives can meet only in the points common to the intersection and to the plane of the picture; they will therefore be parallel to each other.

Hence we derive a very simple method of putting lines and points in perspective.

We draw a perpendicular OO″ from the eye to the plane of delineation (*fig.* 44); the point O″ where it meets Fig. 44.
plane is called the *centre of the picture*, sometimes call-

Fig. 44. ed the *point of sight.* It follows from what has been said, that the perspectives of all lines perpendicular to the plane of the picture, must meet in this point.

We project the proposed point P upon the plane of the picture, which we suppose to be vertical; the point P″ where this projection falls, will be that in which the perpendicular drawn from the proposed point to the plane of the picture, meets it; and P″O″ will be the perspective of this line.

Next draw P′*M*, making with AB an angle equal to half a right angle; this will be the projection of a horizontal line drawn from the point P to the plane of the picture and making the same angle with it; and its meeting with this plane will be at the point M″ placed at a height *M*M″ equal to P′P. But if we take, upon O″D″ parallel to AB, a magnitude O″D″ equal to the distance OO″ of the eye from the picture, it is evident that the line OD″ will be parallel to all the horizontal lines which can be drawn to the picture, at an inclination of 45°, in the direction *M*P′; consequently the perspectives of these lines must all meet in the point D″, which is called the *point of distance.* Having drawn M″D″, this straight line must contain the perspective of the point P; but this perspective must also be found ~~under~~ O″P″; it is therefore in R″.

From the three preceding articles we derive the following rules.

(1). *When an original straight line is parallel to the base of the picture, the perspective of this line is also parallel to the base of the picture* (103).

(2). *The perspectives of all straight lines perpendicular to the picture, are directed towards the* centre of the picture (102).

(3). *The perspectives of all horizontal straight lines which make with the picture an angle of* 45°, *will meet in the point of distance* (103).

(4). *The perspective of a horizontal straight line making any angle whatever with the picture, will have its vanishing point in the horizon, at the intersection with the picture of a straight line drawn parallel to the proposed line, through the eye.*

(5). *The horizon of the picture is the* vanishing line *of the perspectives of all horizontal planes.*

(6). And by analogy—*The vertical line passing*

*through the centre of the picture, is the vanishing line of all vertical planes perpendicular to the plane of the picture.*

(7). *The perspectives of planes parallel to the picture, can have no vanishing lines, and are always figures similar to their originals.*

**104.** We can find the perspective of objects by means of the *receding scale*, which dispenses with tracing their ground plan, and elevation; it is constructed in the following manner.

We refer the objects to three rectangular co-ordinate planes, the first horizontal, and passing through the ground line AB (*fig.* 45); the second vertical, perpendicular to the plane of delineation, and passing through the edge BT; the third, the plane of delineation itself, ABT, which we here suppose to be vertical. A point will then be given when we know its respective distances from these three planes (30). The distance of this point from the plane of the picture will be measured upon BC; its distance from the vertical plane passing through BC and BT, will be measured upon AB; and its distance from the horizontal plane, or the height of the point, will be measured upon BT. Now, the two lines AB and BT being in the plane of the picture, it is sufficient to transfer to this plane the divisions of the third line, BC; this is done by drawing to the centre of the picture the line BO″ which will be the perspective of the line BC, and by drawing to the point of distance D″, the straight lines *a*D″, 1 D″, 2 D″ &c. which will cut BO″ in the points *c*, 1, 2, 3, &c. corresponding to the parts B *b*, *b* 1, 1 2, &c. of the line BC. Fig. 45.

The line BO″, thus divided, is the *receding scale* which marks the apparent *sinking* of objects in the picture; and if we draw through the points of division, of this scale, lines parallel to AB, they may be considered as the *ground lines* of several planes drawn parallel to the plane of the picture at the *depths* marked by the corresponding divisions of the scale; they would contain the perspectives of horizontal projections, or the *bases* of objects situated in these planes.

If we then take upon the straight line AB, which is called *the front scale*, a part B *e* equal to the distance of the proposed point from the vertical plane passing through

Fig. 45. BC and BT, and draw to the centre of the picture the straight line $e\,O''$, the meeting of this line with $g\,2$ parallel to AB, will give the perspective of the horizontal projection, or of the *base* of the proposed object.

Finally, if upon BT, *the scale of heights*, we take the part $e\,f$ equal to the height of the proposed point, and draw $f\,O''$, this last straight line will meet $g\,h$ perpendicular to $g\,2$ at the point $h$ which will be the perspective sought.

We see that this process gives, by operating directly upon the plane of delineation, the perspective of all the objects which we may wish to represent, when we have constructed the *receding scale.*

105. When the picture is so large as to render the construction inconvenient, we may calculate the divisions of the receding scale, by considering the similar triangles $O''\,c\,D''$ and $a\,c\,B$; from which we have

$$\frac{a\text{B}}{\text{B}c}=\frac{O''D''}{O''c},\ \frac{a\text{B}+O''D''}{\text{B}c+O''c}=\frac{a\text{B}}{\text{B}c},\ \frac{a\text{B}+O''D''}{\text{B}O''}=\frac{a\text{B}}{\text{B}c}.$$

The division of this scale gives the distances of straight lines which represent the base lines of perspective planes parallel to the picture. The heights $h\,g$ are also calculated by a similar proportion, since $\frac{e\,f}{h\,g}=\frac{e\,O''}{g\,O''}$, and the straight lines $e\,O''$ and $g\,O''$ are evidently to each other as the distances $\text{B}O''$ and $O''\,2$.

The proportion $\frac{a\text{B}}{\text{B}c}=\frac{O''D''}{O''c}$ will give the distance $O''D''$ of the eye from the picture, when we have given the straight line $\text{B}O''$, the space $a\text{B}$, and its perspective $\text{B}\,c$.

106. *General Remark.* We have, in what precedes, the general means of putting into perspective the apparent outlines and the remarkable points of objects; but these processes, which constitute *linear perspective,* are by no means sufficient to give a complete representation of a body. Light and shade, and the gradations of tint, all concur to represent the prominences, depressions, and distances of objects. All these circumstances may be rigorously determined by methods analogous to those which have been given. To do this, it is only necessary to make such an analysis of the enunciation of

the question as will enable us to discover the mathematical conditions upon which it depends.

With respect to shadows, for example, if the luminous body is reduced to a point, it is evident that they will be determined by the space comprehended by a conical surface tangent to the opaque body, and having for its summit the luminous point. Consequently, to determine the shadow thrown upon any surface whatever, is to find the portion which this cone embraces of the surface in question, that is, the portion circumscribed by the curve which is the intersection of this cone with the proposed surface.

We cannot here discuss those considerations which require knowledge foreign from Geometry; they are mentioned only to show of what use, in the arts, may be a familiarity with the geometry of space.

## *Spherical Projections.*

**107.** In spherical representations for geographical, astronomical, and nautical purposes, a perspective of the outline with other remarkable lines and points in the body is made upon a plane supposed to pass through the centre of the sphere and perpendicular to a straight line drawn through this centre to the eye. This section of the sphere by the plane of delineation is called *the primitive circle* of the perspective projection. The plane itself is called the *primative plane.* For these purposes two kinds of projections are used, the *orthographic* and the *stereographic.*

**108.** *Orthographic Projection.* In the orthographic projection the eye is supposed to be situated at an infinite distance from the body to be projected; so that the rays of light which proceed to the eye from the various points in the body, may be considered as making an infinitely small angle with each other; that is, they are parallel. And the sphere being transparent as well as the plane of delineation, these rays will project orthographically upon this plane, the various lines and points belonging to the farther hemisphere.

**108.** As *all great circles perpendicular to the plane of the perspective,* are in planes passing through the eye,

they *will be projected in straight lines drawn through the centre of the primitive.* And *every small circle perpendicular to the primitive, will have for its projection, that chord of the primitive which results from the intersection of the plane of this circle with the plane of the primitive,* (76).

109. *All circles of the sphere, which are parallel to the primitive, have for their projections, circles equal to themselves;* as the visual rays proceeding from the original circle to the eye, form a cylinder of which the circle and its perspective are parallel sections, (El. 275).

110. [*The orthographic projections of a circle inclined to the primitive, will be an ellipse* (77) *whose transverse axis is equal to the diameter of the circle, and whose conjugate axis is equal to this diameter multiplied by the cosine of the inclination.* If the circle be a great circle, the centre of the eliptical projection will be in the centre of the primitive.

Any ordinate of the ellipse will be equal to the corresponding semi-chord of the circle multiplied by the cosine of inclination.]

111. Problem. *To find the conjugate axis of the orthographic projection of an inclined great circle.*

Let the arc M m″ measure the inclination of the pro-
Fig. 46. posed circle to the plane of the primitive (*fig.* 46); draw m″m parallel to the horizontal diameter of the primitive (which is also the transverse axis of the ellipse) till it meets the vertical diameter in m; take O m′ equal to O m, and m m′ will be the conjugate axis required.

To find the ordinate o n, corresponding to the semichord o N; make the angle N o n″ equal to the inclination of the proposed circle to the plane of the primitive; take o n″ equal to o N, and draw n″n parallel to A o; o n will be the *ordinate* required. By finding a sufficient number of ordinates the curve may be approximately traced.

112. *Stereographic Projection.* The stereographic projection of a sphere is a perspective in which the eye is situated at the *pole* of the primitive circle. The sphere and the plane of delineation being here also supposed to be transparent, the various lines and points upon the farther hemisphere, will be projected upon the primitive

circle by the visual rays passing from the several points Fig. 46.
to the eye; and the lines and points of the nearer hemisphere will be projected upon the same plane *beyond* the circumference of the primitive.

**113.** *All great circles perpendicular to the primitive*, as they pass through the axis of the primitive, will have their planes pass through the eye, and *will* therefore *be projected in straight lines passing through the centre of the primitive.*

**114.** *Every circle parallel to the primitive will have a circle for its projection;* for the visual rays, in this case, constitute a right cone whose base is the original circle and of which the projection is a section parallel to the base (El. 268).

As the axis of this cone of rays passes through the centre of the primitive, *The centre of the projection of circles parallel to the primitive, will be in the centre of the primitive circle.*

**115.** PROBLEM. *To find the distance from the centre of the primitive, of the projection of any point whose distance from the farther pole is given.*

Let AP′BO′ (*fig.* 47) be the horizontal projection of the Fig. 47
sphere; AB the horizontal trace of the plane of delineation, or the horizontal projection of the primitive circle; and M′ the horizontal projection of the proposed point (supposed in the circumference of the horizontal great circle), O′ being the horizontal projection of the eye. Draw O′M′ meeting AB in the point m; and C′m will be the distance of the perspective of the proposed point from the centre of the primitive circle.

**116.** [The angle P′O′M′ is measured by half the arc P′M′ (El. 116); and by drawing the arc C′E it will be perceived that C′m is the tangent of this angle, that is, the tangent of half the arc P′M′, which measures the distance of the point M from the farther pole and is called the *polar distance* of the point. We therefore say,— *The projection of every point in the surface of a sphere, is at a distance from the centre of the primitive circle, equal to the tangent of half the polar distance of that point.*

**117.** This gives us the radius of the projection of every circle parallel to the primitive; for every part of its circumference being at the same distance from the

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Fig. 47. pole, *The tangent of half the polar distance of a circle parallel to the primitive, is the radius of the projection of that circle.*]

118. PROBLEM. *To find the projection of a circle inclined to the plane of the primitive.*

Let the circle in question be a small circle; the visual rays proceeding to the eye from the several points in this circle will constitute a scalene cone. The intersection of this cone with the plane of the primitive will be the projection required. Suppose the sphere to be so situated that the plane determined by the axis of the cone and the axis of the primitive circle shall be horizontal;
Fig. 48. and let figure 48 represent the horizontal projection of the sphere, AB being the horizontal trace of the plane of the picture, and M′N′ the horizontal projection of the circle proposed.

The angle n m O′, made by the two chords AB and O′M′, is measured by half the sum of the arcs AO′, BM′, (El. 115) or (as AO′ is equal to BO′) by half of the arc O′M′; but this is the measure of the angle O′N′M′ (El. 116), therefore the angles n m O′ and M′N′O′ are equal. The angle N′O′M′ is common to the two triangles n m O′ and N′M′O′; and having two angles of the one equal to two angles of the other, the other angles must be equal, that is, the angle O′M′N′ is equal to the angle O′ n m; and this intersection of the cone of rays by the perspective plane, is a *sub-contrary* section, and is therefore *a circle*, the base of the scalene cone being a circle.

To show that this section is a circle, suppose a section parallel to this, made by a plane whose horizontal trace is n′ m′, and whose intersection with the base of the cone is a vertical line having its horizontal projection in the point *c*′. This line is a common chord of the two curvilinear sections of this cone; and as the section whose horizontal projection is M′N′ is a circle, the square of half this common chord is equal to the product of the two segments of the diameter; that is, equal to *c*′ N′ × *c*′ M′, (El. 126).

The angle *c*′ m′ M′ has its sides parallel to those of the angle *c* m O′ which was shown to be equal to *c*′ N′ n′; therefore the angle *c*′ m′ M′ is equal to *c*′ N′ n′; and as the two angles at *c*′ are vertical angles, the two triangles are equiangular and therefore similar, and give the pro-

portion $\frac{c'N'}{c'n'} = \frac{c'm'}{c'M'}$; and, by multiplying by the denominators, we obtain $c'N' \times c'M' = c'm' \times c'n'$. This last product, therefore, equals the square of half the common chord of the curves; this last curve is consequently a circle; and as this is a section parallel to the perspective in question, the perspective is therefore a circle. And we say,—*The stereographic projection of a small circle inclined to the plane of the primitive, is also a circle.* Fig. 48.

119. If the circle in question were a great circle, the line M'N' would pass through the point C'; and a process similar to the above would show, that the intersection by the plane of the primitive circle, of a cone of which this is the base and O the summit, would be a circle. We thence conclude that,—*The stereographic projections of all circles of the sphere are circles.*

120. As the process in article 118, gives us the two extremities m and n of the diameter of the projection of the inclined circle proposed; $c$ the middle of m n will be the centre of the projection, and $c$ m the radius. We can therefore find the projection of any circle inclined to the plane of the primitive.

121. If the small circle be perpendicular to the primitive, so that its horizontal projection may be EF (*fig.* 47); the polar distance O'E or P'F will give two points of Fig. 47.
the projection, E and F; and by drawing the straight line O'F we obtain the distance $g$ C' from the centre of the picture to the nearest point of the curve; which take from the centre of the primitive, on the horizontal line. Having now *three* points in a circular curve, the curve is readily drawn, (El. 110).

122. We have, in figure 49, an orthographic projection of the sphere upon the plane of the equator. The meridians, being perpendicular to the plane of the projection, are straight lines; the parallels of latitude are circles concentric with the primitive. Fig. 49.

In figure 50 we have a stereographic projection of the same. The meridians, as they pass through the eye, are projected in straight lines; all the other circular lines upon the sphere have circular curves for their projections. Fig. 50.

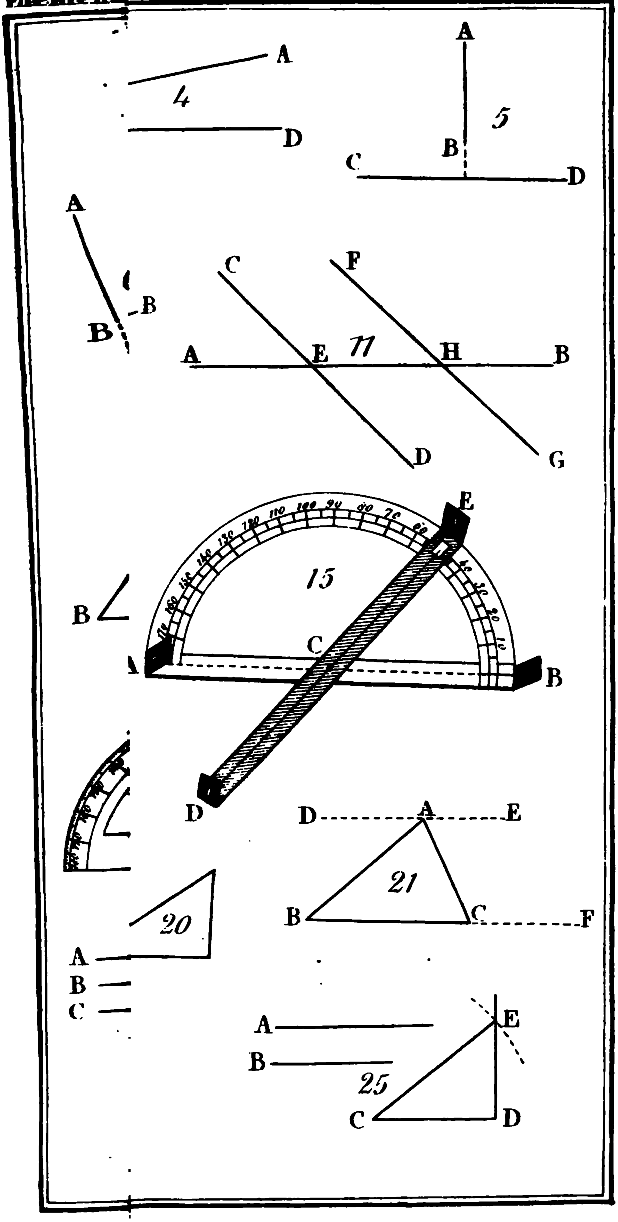
Fig. 51. 123. Figure 51 exhibits an orthographic projection of the sphere upon the plane of a meridian; this gives all the other meridians elliptical curves, and all the parallels
Fig. 52. of latitude straight lines. In figure 52 we have a stereographic projection of the sphere upon the plane of a meridian. As the eye is supposed to be at the intersection of one of the meridians represented and the equator, the projections of these are straight lines. All the other circles are projected in circular curves.

*Remark.* It will be perceived that the several portions of a spherical surface are not represented in their proportional magnitudes by either of these projections; the orthographic having the parts most crowded near the circumference of the primitive; and the stereographic being most crowded near the centre of the primitive.

These projections, notwithstanding these imperfections, serve for the representation of astronomical phenomena, and the construction of astronomical and nautical problems. But in those geographical representations in which a hemisphere is to be exhibited at once, a method called *globular projection* is generally used. This is not indeed a projection; it is a construction, made by dividing the diameter which represents the equator, into equal parts to represent the same number of degrees of longitude, and drawing meridian circles through these divisions and the poles; and by dividing the polar diameter into equal parts, and also the semicircles on each side of it into the same number of equal parts, and drawing circular curves through the corresponding divisions
Fig. 53. to represent the parallels of latitude, (*fig.* 53).

There are other methods of representing portions of the earth's surface, but we cannot go farther in this subject. Our object has been to give merely an *introduction* to Descriptive Geometry; a sketch or outline of some of the more practical, and thence more generally interesting subjects, which this science instructs us to discuss.

THE END.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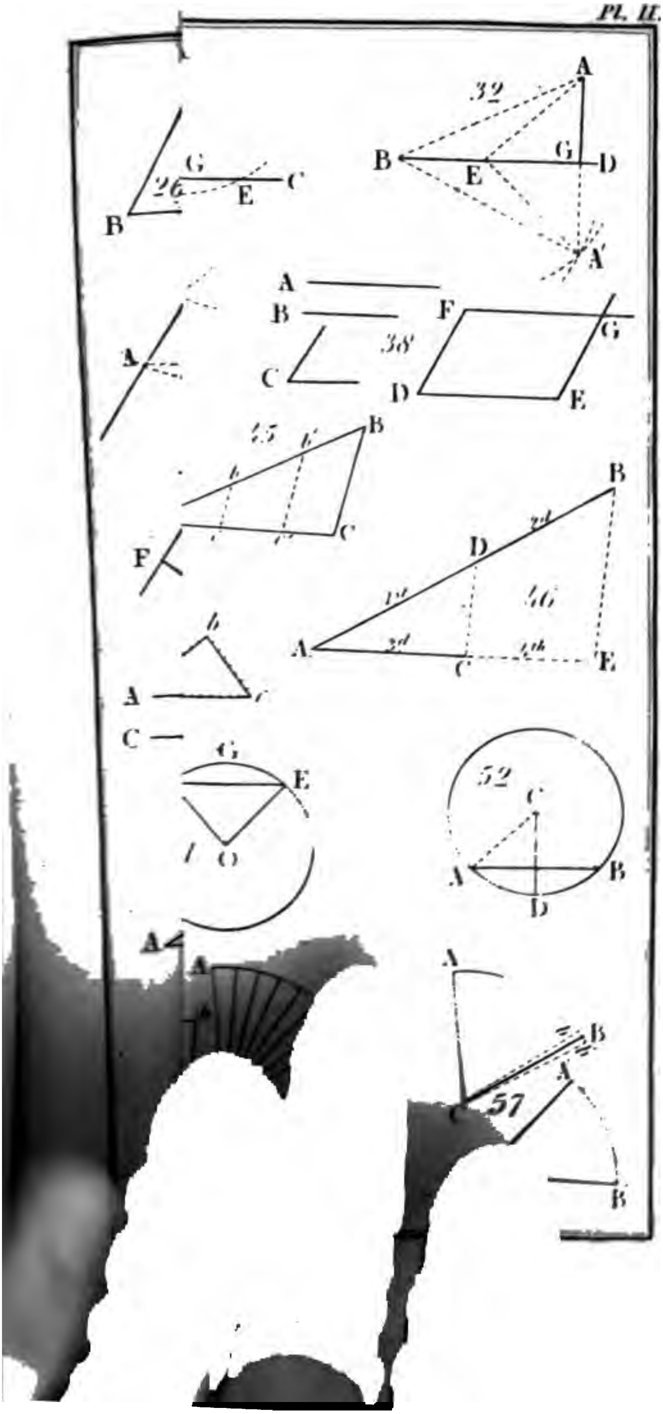
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Pl. IV.